Volume Viscosity and Internal Energy Relaxation : Symmetrization and Chapman-Enskog Expansion

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Abstract

We analyze a mathematical model for the relaxation of translational and internal temperatures in a nonequilibrium gas. The system of partial differential equations—derived from the kinetic theory of gases—is recast in its natural entropic symmetric form as well as in a convenient hyperbolic-parabolic symmetric form. We investigate the Chapman-Enskog expansion in the fast relaxation limit and establish that the temperature difference becomes asymptotically proportional to the divergence of the velocity field. This asymptotic behavior yields the volume viscosity term of the limiting one-temperature fluid model.

1 Introduction

The relaxation of internal energy is of fundamental importance in fluids with internal degrees of freedom and is related to the concept of volume viscosity as established by the kinetic theory of polyatomic gases [27, 46, 6, 16, 37, 38, 3, 4, 5] and statistical thermodynamics of nonequilibrium processes [34]. The kinetic theory [6, 16, 11, 12, 37, 3] as well as experimental measurements [39, 43] have further confirmed that the volume viscosity coefficient is of the same order as the shear viscosity coefficient for polyatomic gases and its impact in fluid mechanics—especially for fast flows—has also been established [29, 9, 10, 26, 1, 3]. This is a strong motivation for investigating mathematically the fast relaxation of internal energy as well as the concept of volume viscosity in nonequilibrium fluid models. With this aim in mind, we study in this paper a mathematical model for the relaxation of translational and internal temperatures in a nonequilibrium gas. Symmetrization of the corresponding system of partial differential equations is obtained as well as a convenient hyperbolic-parabolic form. The Chapman-Enskog expansion is performed in the fast relaxation limit and yields the volume viscosity coefficient. The convergence analysis for small relaxation times lay out of the scope of the present work and will be presented in a future paper [24].

The system of partial differential equations modeling fluids out of thermodynamic equilibrium as derived from the kinetic theory of gases is first presented [3, 4]. The derivation of the model from the multitemperature kinetic theory of polyatomic gases in summarized in Appendix A. The resulting equations may be split between conservation equations, thermodynamics, energy exchange rate and transport fluxes. The nonequilibrium fluid is characterized by two temperatures, one associated with the translational degrees of freedom and another one associated with the internal degrees of freedom. The physical entropy is the sum of an entropy of translational origin and an entropy of internal origin. The fluid model out of thermodynamic equilibrium is shown to satisfy the second principle of thermodynamics, that is, entropy production due to transport fluxes and energy exchanges are shown to be nonnegative. We further introduce the corresponding local equilibrium temperature and summarize the traditional physical derivation of the volume viscosity term. In a relaxation regime, when energy exchanges are fast, the temperature difference becomes approximately proportional to the divergence of the velocity field and this leads to the volume viscosity term of the one-temperature limit fluid model. One of the goal of this paper is to justify rigorously such an approximation.

We next investigate symmetrized forms for the system of partial differential equations. Symmetrized forms are important for analyzing hyperbolic as well as hyperbolic-parabolic systems of partial differential equations modeling fluids [25, 17, 45, 30, 32, 7, 40, 19, 18, 8, 49, 15, 50, 33, 2, 41]. They are useful for a priori estimates, existence theorems [45, 30, 44, 19] as well as finite element formulations [28]. Existence of a symmetrized form is related to the existence of a mathematical entropy compatible with convective terms, dissipative terms and relaxation of energy. We explicitly evaluate the natural entropic symmetrized form for fluids out of thermodynamic equilibrium with a mathematical entropy taken to be the opposite of the physical entropy per unit volume.

We then investigate normal forms, that is, symmetric hyperbolic-parabolic composite forms of the system of partial differential equations [30, 32]. The symmetrizing normal variable w must be chosen carefully in order that the 'mass matrix' $\overline{\mathsf{A}}_0$ in front of the time derivative operator ∂_t leaves invariant the fast manifold or equivalently commutes with the orthogonal projector π on the fast manifold. To this aim, we notably use the local equilibrium temperature in the definition of the normal variable. In addition, the source term is naturally in quasilinear form as is typical in a relaxation framework and often encountered in mathematical physics [51]. We also introduce conditions which guarantee that the structure of the source term of the natural entropic symmetrized form $\widetilde{\Omega}$ is transported to that of the normal form $\overline{\Omega}$. The mathematical framework needed to investigate the fast relaxation limit is completed by introducing the small parameter ϵ associated with energy relaxation and the small parameter ϵ_d associated with second order dissipative terms.

We further present the system of partial differential equations modeling fluid at thermodynamic equilibrium. This system is the limiting set of equations obtained as the internal energy relaxation time goes to zero. Symmetrized forms for the corresponding system of equation are also obtained and their links with those of the nonequilibrium model.

We finally investigate the Chapman-Enskog expansion for hyperbolic-parabolic systems of partial differential equations extending previous work on hyperbolic systems [7, 50]. The resulting first order accurate governing equations for the slow variable then involves dissipative coefficients arising from perturbed convective terms as well as inherited directly from the system out of equilibrium. We simplify the expression of the former dissipative coefficients by using convenient generalized inverses, investigate their smoothness, and study the symmetrizing properties of the resulting systems of partial differential equations. Applying these results to the fast internal energy relaxation problem, we establish that the volume viscosity term of the one-temperature fluid model arises through a Chapman-Enskog expansion. Both the volume viscosity term arising from the perturbed convective fluxes and the shear viscosity term inherited from the out of equilibrium viscous tensor are finally involved in the equilibrium viscous tensor. This yields a rigorous framework for the asymptotic fast relaxation limit more satisfactory than traditional physically intuitive procedures. The general relation between the volume viscosity coefficient and internal energy relaxation time is also recovered.

The equations governing one-temperature fluids may thus be derived through a double Chapman-Enskog expansion, first from a kinetic framework to a fluid model out of thermodynamic equilibrium, and then from the non equilibrium model towards the equilibrium one-temperature fluid. They may also be obtained, however, through a direct Chapman-Enskog expansion going directly from the kinetic model to the equilibrium one-temperature fluid. Differences and similarities between the resulting sets of partial differential equations are addressed.

The nonequilibrium two-temperature model is presented in Section 2 and its symmetrization is obtained in Section 3. The corresponding equations at equilibrium are examined in Section 4. The Chapman-Enskog expansion is addressed in Section 5 and the kinetic derivation of the model is sketched in Appendix A.

2 Governing equations

The system of equations modeling fluids out of thermodynamic equilibrium derived from the kinetic of gases [3, 4] is investigated. The mathematical assumptions are presented and the system of partial differential equations is rewritten in quasilinear form. The local thermal equilibrium temperature is introduced and the traditional physical derivation of the volume viscosity term is presented.

2.1 Conservation equations

In a nonequilibrium gas with internal degrees of freedom, the conservation of mass, momentum, internal energy and total energy may be written in the form [3, 4]

$$\partial_t \rho + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0, \tag{2.1}$$

$$\partial_t(\rho \boldsymbol{v}) + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}\otimes\boldsymbol{v} + p\boldsymbol{I}) + \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = 0, \tag{2.2}$$

$$\partial_t \mathcal{E}_{\mathrm{in}} + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\mathcal{E}_{\mathrm{in}}) + \boldsymbol{\nabla}\cdot\boldsymbol{Q}_{\mathrm{in}} = \omega_{\mathrm{in}}, \tag{2.3}$$

$$\begin{aligned}\partial_t(\mathcal{E}_{\mathrm{tr}} + \mathcal{E}_{\mathrm{in}} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2) + \boldsymbol{\nabla}\cdot\big(\boldsymbol{v}(\mathcal{E}_{\mathrm{tr}} + \mathcal{E}_{\mathrm{in}} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2 + p)\big)&\\ + \boldsymbol{\nabla}\cdot(\boldsymbol{Q}_{\mathrm{tr}} + \boldsymbol{Q}_{\mathrm{in}} + \boldsymbol{\Pi}\cdot\boldsymbol{v}) = 0.&\end{aligned} \tag{2.4}$$

In these equations ∂_t denotes the time derivative operator, $\boldsymbol{\nabla}$ the space derivative operator, ρ the mass density, $\boldsymbol{v}$ the fluid velocity, $\otimes$ the tensor product symbol, p the pressure, $\boldsymbol{\Pi}$ the viscous tensor, $\boldsymbol{I}$ the unit tensor in the physical space $\mathbb{R}^d$, $\mathcal{E}_{\mathrm{in}}$ the internal energy of internal origin per unit volume, $\boldsymbol{Q}_{\mathrm{in}}$ the heat flux of internal origin, ω_{in} the energy exchange rate, $\mathcal{E}_{\mathrm{tr}}$ the internal energy of translational origin per unit volume, and $\boldsymbol{Q}_{\mathrm{tr}}$ the heat flux of translational origin. We denote by $d \geq 1$ the dimension of the physical model under consideration so that we have $\boldsymbol{v} = (v_1, \dots, v_d)^t$ and $\boldsymbol{\nabla} = (\partial_1, \dots, \partial_d)^t$ with v_i denoting the velocity in the ith spatial direction, ∂_i the derivation in the ith spatial direction and bold symbols are used for vector or tensor quantities in the physical space $\mathbb{R}^d$. These equations (2.2)–(2.4) may further be combined in order to form an evolution equation for the internal energy of translational origin

$$\partial_t \mathcal{E}_{\mathrm{tr}} + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\mathcal{E}_{\mathrm{tr}}) + \boldsymbol{\nabla}\cdot\boldsymbol{Q}_{\mathrm{tr}} = -p\boldsymbol{\nabla}\cdot\boldsymbol{v} - \boldsymbol{\Pi}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{v} - \omega_{\mathrm{in}}. \tag{2.5}$$

The equations (2.2)–(2.4) have to be completed by relations expressing the thermodynamic properties $\mathcal{E}_{\mathrm{in}}$, $\mathcal{E}_{\mathrm{tr}}$, and p, the rate of energy exchange ω_{in}, and the transport fluxes $\boldsymbol{\Pi}$, $\boldsymbol{Q}_{\mathrm{in}}$ and $\boldsymbol{Q}_{\mathrm{tr}}$.

2.2 Thermodynamics

The pressure p, the total internal energy per unit volume $\mathcal{E}$, the internal energy of translational origin per unit volume $\mathcal{E}_{\mathrm{tr}}$, and the internal energy of internal origin per unit volume $\mathcal{E}_{\mathrm{in}}$ are in the form

$$p = \rho r T_{\mathrm{tr}}, \qquad \mathcal{E} = \rho e, \qquad \mathcal{E}_{\mathrm{tr}} = \rho e_{\mathrm{tr}}, \qquad \mathcal{E}_{\mathrm{in}} = \rho e_{\mathrm{in}}, \tag{2.6}$$

where r denotes the gas constant per unit mass, e the total internal energy per unit mass, e_{tr} the internal energy of translational origin per unit mass, and e_{in} the internal energy of internal origin per unit mass. These energies per unit mass $e(T_{\mathrm{tr}}, T_{\mathrm{in}})$, $e_{\mathrm{tr}}(T_{\mathrm{tr}})$, and $e_{\mathrm{in}}(T_{\mathrm{in}})$ are given by

$$e = e_{\mathrm{tr}} + e_{\mathrm{in}}, \qquad e_{\mathrm{tr}} = c_{\mathrm{v,tr}} T_{\mathrm{tr}}, \qquad e_{\mathrm{in}} = e_{\mathrm{in,st}} + \int_{T_{\mathrm{st}}}^{T_{\mathrm{in}}} c_{\mathrm{in}}(\theta)\,\mathrm{d}\theta, \tag{2.7}$$

where $c_{\mathrm{v,tr}} = \frac{3}{2}r$ denotes the translational heat at constant volume per unit mass, T_{tr} the translational temperature, c_{in} the internal heat per unit mass, T_{in} the internal temperature, T_{st} the standard temperature, and $e_{\mathrm{in,st}}$ the internal formation energy at the standard temperature. We will also use in the following the translational heat at constant pressure per unit mass $c_{p,\mathrm{tr}} = \frac{5}{2}r$ and the formation energy at zero temperature $e_{\mathrm{in}}^0 = e_{\mathrm{in}}(0)$.

The total entropy per unit volume $\mathcal{S}$, the translational entropy per unit volume $\mathcal{S}_{\mathrm{tr}}$ and the internal entropy per unit volume $\mathcal{S}_{\mathrm{in}}$ are defined by

$$\mathcal{S} = \rho s, \qquad \mathcal{S}_{\mathrm{tr}} = \rho s_{\mathrm{tr}}, \qquad \mathcal{S}_{\mathrm{in}} = \rho s_{\mathrm{in}}, \tag{2.8}$$

where s denotes the total entropy per unit mass, s_{tr} the translational entropy per unit mass, and s_{in} the internal entropy per unit mass. These entropies per unit mass $s(\rho, T_{\mathrm{tr}}, T_{\mathrm{in}})$, $s_{\mathrm{tr}}(\rho, T_{\mathrm{tr}})$, and $s_{\mathrm{in}}(T_{\mathrm{in}})$ are in the form

$$s = s_{\mathrm{tr}} + s_{\mathrm{in}}, \quad s_{\mathrm{tr}} = s_{\mathrm{tr,st}} + c_{\mathrm{v,tr}} \log\Big(\frac{T_{\mathrm{tr}}}{T_{\mathrm{st}}}\Big) - r\log\Big(\frac{\rho}{\rho_{\mathrm{st}}}\Big), \quad s_{\mathrm{in}} = s_{\mathrm{in,st}} + \int_{T_{\mathrm{st}}}^{T_{\mathrm{in}}} \frac{c_{\mathrm{in}}(\theta)}{\theta}\,\mathrm{d}\theta, \tag{2.9}$$

where $s_{\rm tr,st}$ denotes the translational formation entropy at the standard temperature $T_{\rm st}$ and pressure $p_{\rm st}$, $\rho_{\rm st} = m p_{\rm st}/(RT_{\rm st})$ the standard mass density, m the molar mass of the gas, and $s_{\rm in,st}$ the internal formation entropy at the standard temperature $T_{\rm st}$ and pressure $p_{\rm st}$. From the definition of $s_{\rm tr}$ and $s_{\rm in}$ it is further obtained that $T_{\rm tr}{\rm d}s_{\rm tr} = {\rm d}e_{\rm tr} - (p/\rho^2){\rm d}\rho$ and $T_{\rm in}{\rm d}s_{\rm in} = {\rm d}e_{\rm in}$ where d denotes the total differential, so that the Gibbs relation out of thermodynamic equilibrium may be written [3, 4]

$$
{\rm d}s = \frac{c_{\rm v,tr}}{T_{\rm tr}}{\rm d}T_{\rm tr} + \frac{c_{\rm in}}{T_{\rm in}}{\rm d}T_{\rm in} - \frac{r}{\rho}{\rm d}\rho. \tag{2.10}
$$

From the differential of entropy (2.10), after some algebra, the following entropy governing equation is obtained [3, 4]

$$
\partial_t \mathcal{S} + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\mathcal{S}) + \boldsymbol{\nabla}\cdot\Big(\frac{\boldsymbol{Q}_{\rm tr}}{T_{\rm tr}} + \frac{\boldsymbol{Q}_{\rm in}}{T_{\rm in}}\Big) = \mathfrak{v}, \tag{2.11}
$$

where the entropy production $\mathfrak{v}$ given by

$$
\mathfrak{v} = -\frac{\boldsymbol{Q}_{\rm tr}\cdot\boldsymbol{\nabla}T_{\rm tr}}{T_{\rm tr}^2} - \frac{\boldsymbol{Q}_{\rm in}\cdot\boldsymbol{\nabla}T_{\rm in}}{T_{\rm in}^2} - \frac{\boldsymbol{\Pi}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{v}}{T_{\rm tr}} + \frac{\omega_{\rm in}(T_{\rm tr}-T_{\rm in})}{T_{\rm tr}T_{\rm in}}. \tag{2.12}
$$

We will later establish that both the entropy production due to variables' gradients and the entropy production due to energy exchange are nonnegative. For future use, we also introduce the translational and internal Gibbs functions per unit mass $g_{\rm tr} = e_{\rm tr} + rT_{\rm tr} - T_{\rm tr}s_{\rm tr}$ and $g_{\rm in} = e_{\rm in} - T_{\rm in}s_{\rm in}$ as well as the translational enthalpy per unit mass $h_{\rm tr} = e_{\rm tr} + rT_{\rm tr}$.

The rate of energy exchange between the translational and internal degrees of freedom $\omega_{\rm in}$ may finally be written [3]

$$
\omega_{\rm in} = \frac{\rho c_{\rm in}}{\tau_{\rm in}}(T_{\rm tr} - T_{\rm in}), \tag{2.13}
$$

where $\tau_{\rm in}$ denotes the energy exchange time.

2.3 Transport fluxes

In the framework of the kinetic theory of polyatomic gases out of thermodynamic equilibrium, the translational and internal heat fluxes are in the form [3]

$$
\boldsymbol{Q}_{\rm tr} = -\,\lambda_{\rm tr,tr}\boldsymbol{\nabla}T_{\rm tr} - \lambda_{\rm tr,in}\boldsymbol{\nabla}T_{\rm in}, \tag{2.14}
$$

$$
\boldsymbol{Q}_{\rm in} = -\,\lambda_{\rm in,tr}\boldsymbol{\nabla}T_{\rm tr} - \lambda_{\rm in,in}\boldsymbol{\nabla}T_{\rm in}, \tag{2.15}
$$

where $\lambda_{\rm tr,tr}$, $\lambda_{\rm tr,in}$, $\lambda_{\rm in,tr}$, and $\lambda_{\rm in,in}$ denote thermal conductivities. Both temperature gradients are involved in both fluxes even though the cross thermal conductivities $\lambda_{\rm tr,in}$ and $\lambda_{\rm in,tr}$ are generally smaller than $\lambda_{\rm tr,tr}$ and $\lambda_{\rm in,in}$ [3]. We will denote by $\boldsymbol{Q}_{\rm tr} = (Q_{{\rm tr},1},\dots,Q_{{\rm tr},d})^t$ and $\boldsymbol{Q}_{\rm in} = (Q_{{\rm in},1},\dots,Q_{{\rm in},d})^t$ the spatial components of the heat fluxes $\boldsymbol{Q}_{\rm tr}$ and $\boldsymbol{Q}_{\rm in}$.

On the other hand, the viscous tensor is given by

$$
\boldsymbol{\Pi} = -\eta\big(\boldsymbol{\nabla}\boldsymbol{v} + (\boldsymbol{\nabla}\boldsymbol{v})^t - \tfrac{2}{d'}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{I}\big), \tag{2.16}
$$

where η denotes the shear viscosity and d' the dimension of the velocity space in the underlying kinetic framework. It will be assumed in the following that the dimension of the kinetic velocity space d' is such that $2 \leq d'$ and $d \leq d'$. The assumption $1 \leq d \leq d'$ means that the spatial dimension d of the model has eventually been reduced, so that the equations are considered in $\mathbb{R}^d$ independently from kinetic velocity fluctuations which always have the maximum dimension d'. The assumption $2 \leq d'$ is natural since $d' = 3$ in our physical world and since $\boldsymbol{\Pi}$ is identically zero when $d' = 1$. We also denote by Π_{ij}, $1 \leq i,j \leq d$, the components of the viscous tensor $\boldsymbol{\Pi}$.

The structure of these transport fluxes derived from the kinetic theory of non equilibrium gases is naturally compatible with the Curie principle. The thermal conductivities $\lambda_{\rm tr,tr}$, $\lambda_{\rm tr,in}$, $\lambda_{\rm in,tr}$, and $\lambda_{\rm in,in}$ and the shear viscosity η are defined in a kinetic framework in terms of bracket products between solutions of integral linearized Boltzmann equation [3]. The mathematical structure of the transport coefficients presented in the following is extracted from these kinetic relations. From the expression (2.16) it is also noted that the viscous tensor $\boldsymbol{\Pi}$ does not present a volume viscosity term and our aim is to investigate the apparition of such a contribution in the one-temperature equilibrium limit model as the relaxation time $\tau_{\rm in}$ goes to zero.

2.4 Mathematical assumptions

The mathematical assumptions naturally associated with fluids out of thermodynamic equilibrium are presented. The rescaled energy exchange time $\overline{\tau}_{\mathrm{in}}$ as well as the rescaled transport coefficients $\overline{\eta}$, $\overline{\lambda}_{\mathrm{tr,tr}}$, $\overline{\lambda}_{\mathrm{tr,in}}$, $\overline{\lambda}_{\mathrm{in,tr}}$, and $\overline{\lambda}_{\mathrm{in,in}}$ are introduced in order to investigate the fast relaxation limit. The assumptions associated with thermodynamic properties and the energy exchange rate are the following where $\varkappa \geq 3$ denotes the regularity class of thermodynamic functions [18, 21, 3].

(T_1) *The formation energy $e_{\mathrm{in,st}}$ and formation entropies $s_{\mathrm{tr,st}}$ and $s_{\mathrm{in,st}}$ are real constants. The mass per unit mole m, the gas constant R, and the gas constant per unit mass $r = R/m$ are positive. The internal species heat per unit mass $c_{\mathrm{in}}(T_{\mathrm{in}})$ is a $C^{\varkappa-1}$ function over $[0,\infty)$ and there exist constants $\underline{c}$ and $\overline{c}$ such that $0 < \underline{c} \leqslant c_{\mathrm{in}}(T_{\mathrm{in}}) \leqslant \overline{c}$ for all $T_{\mathrm{in}} \geqslant 0$.*

(T_2) *The energy exchange rate $\tau_{\mathrm{in}}(p, T_{\mathrm{tr}}, T_{\mathrm{in}})$ is in the form*

$$\tau_{\mathrm{in}} = \epsilon \overline{\tau}_{\mathrm{in}} = \epsilon \frac{p^{\mathrm{st}} \overline{\tau}_{\mathrm{in}}^{\mathrm{st}}}{p}, \tag{2.17}$$

where $\epsilon \in (0,1]$ denotes a positive parameter, $\overline{\tau}_{\mathrm{in}}(p, T_{\mathrm{tr}}, T_{\mathrm{in}}) = p^{\mathrm{st}} \overline{\tau}_{\mathrm{in}}^{\mathrm{st}}/p$ the rescaled energy exchange time and $\overline{\tau}_{\mathrm{in}}^{\mathrm{st}}(T_{\mathrm{tr}}, T_{\mathrm{in}})$ the rescaled energy exchange time at the standard pressure p^{st} which only depends on T_{tr} and T_{in}. The rescaled time $\overline{\tau}_{\mathrm{in}}^{\mathrm{st}}$ is a positive $C^{\varkappa}$ function of the two temperatures $T_{\mathrm{tr}}, T_{\mathrm{in}} \in (0,\infty)$.

The extension up to zero temperature of specific heats, energies and enthalpies is commonly used in thermodynamics. The internal specific heat remain bounded away from zero since we consider a polyatomic gas governed by Boltzmann type statistics [3]. This specific heat may also depend on both temperatures but such a dependence would easily be included. The fact that the quantity $p\tau_{\mathrm{in}}$ only depends on $(T_{\mathrm{tr}}, T_{\mathrm{in}})$ is a direct consequence from the kinetic theory and will imply that the volume viscosity only depends on $(T_{\mathrm{tr}}, T_{\mathrm{in}})$. The assumptions associated with the transport coefficients are the following.

(Tr_1) *The coefficients η, $\lambda_{\mathrm{tr,tr}}$, $\lambda_{\mathrm{tr,in}}$, $\lambda_{\mathrm{in,tr}}$, and $\lambda_{\mathrm{in,in}}$ are in the form*

$$\begin{aligned} \eta = \epsilon_d \overline{\eta}, \quad \lambda_{\mathrm{tr,tr}} = \epsilon_d \overline{\lambda}_{\mathrm{tr,tr}}, \quad \lambda_{\mathrm{tr,in}} = \epsilon_d \overline{\lambda}_{\mathrm{tr,in}}, \\ \lambda_{\mathrm{in,tr}} = \epsilon_d \overline{\lambda}_{\mathrm{in,tr}}, \quad \lambda_{\mathrm{in,in}} = \epsilon_d \overline{\lambda}_{\mathrm{in,in}}, \end{aligned} \tag{2.18}$$

where $\epsilon_d \in (0,1]$ denotes a positive parameter, and $\overline{\eta}$, $\overline{\lambda}_{\mathrm{tr,tr}}$, $\overline{\lambda}_{\mathrm{tr,in}}$, $\overline{\lambda}_{\mathrm{in,tr}}$, and $\overline{\lambda}_{\mathrm{in,in}}$ the rescaled transport coefficients. The rescaled coefficients $\overline{\eta}$, $\overline{\lambda}_{\mathrm{tr,tr}}$, $\overline{\lambda}_{\mathrm{tr,in}}$, $\overline{\lambda}_{\mathrm{in,tr}}$, and $\overline{\lambda}_{\mathrm{in,in}}$ are $C^{\varkappa}$ functions of the two temperatures $T_{\mathrm{tr}}, T_{\mathrm{in}} \in (0,\infty)$.

(Tr_2) *For any $T_{\mathrm{tr}}, T_{\mathrm{in}} \in (0,\infty)$, the matrix*

$$\begin{bmatrix} T_{\mathrm{in}}^2 \overline{\lambda}_{\mathrm{in,in}} & T_{\mathrm{tr}}^2 \overline{\lambda}_{\mathrm{in,tr}} \\ T_{\mathrm{in}}^2 \overline{\lambda}_{\mathrm{tr,in}} & T_{\mathrm{tr}}^2 \overline{\lambda}_{\mathrm{tr,tr}} \end{bmatrix}, \tag{2.19}$$

is symmetric positive definite. In the viscous tensor (2.16), the coefficient η is positive and the dimension d' of the kinetic velocity space is such that $\max(2,d) \leq d'$.

These properties are directly deduced from the definition of transport coefficients within the framework of the kinetic theory of nonequilibrium gases [38, 3]. The symmetry properties of the matrix (2.19) may also be interpreted as Onsager type relations associated with the variables $1/T_{\mathrm{in}}$ and $1/T_{\mathrm{tr}}$. We now deduce some properties of the thermal conductivities that will be needed in the following.

Lemma 2.1. *Assuming that (Tr_1) holds, the matrix of thermal conductivities (2.19) is positive definite if and only if $\overline{\lambda}_{\mathrm{tr,in}} > 0$, $\overline{\lambda}_{\mathrm{in,tr}} > 0$ and*

$$|\overline{\lambda}_{\mathrm{tr,in}} \overline{\lambda}_{\mathrm{in,tr}}| < \overline{\lambda}_{\mathrm{tr,tr}} \overline{\lambda}_{\mathrm{in,in}}. \tag{2.20}$$

In this situation, the matrix

$$\begin{bmatrix} T_{\rm in}^2\bar{\lambda}_{\rm in,in} & T_{\rm tr}^2\bar{\lambda}_{\rm in,tr}+T_{\rm in}^2\bar{\lambda}_{\rm in,in} \\ T_{\rm in}^2(\bar{\lambda}_{\rm tr,in}+\bar{\lambda}_{\rm in,in}) & T_{\rm tr}^2(\bar{\lambda}_{\rm tr,tr}+\bar{\lambda}_{\rm in,tr})+T_{\rm in}^2(\bar{\lambda}_{\rm tr,in}+\bar{\lambda}_{\rm in,in}) \end{bmatrix}, \tag{2.21}$$

is also symmetric positive definite.

Proof. Assuming that the matrix (2.19) is positive definite, we obtain that $\lambda_{\rm tr,tr} > 0$, $\lambda_{\rm in,in} > 0$ and that its determinant is positive which yield (2.20) and the converse is straightforward.

On the other hand, both diagonal coefficients of (2.21) are positive and its determinant is evaluated to be that of (2.20) using $T_{\rm in}^2\bar{\lambda}_{\rm tr,in} = T_{\rm tr}^2\bar{\lambda}_{\rm in,tr}$. □

We further deduce from these assumptions that the physical entropy production is nonnegative.

Lemma 2.2. *Assuming that* $(\mathsf{T}_1)(\mathsf{T}_2)$ *and* $(\mathsf{Tr}_1)(\mathsf{Tr}_2)$ *hold, the physical entropy production* (2.12) *in equation* (2.11) *is nonnegative. In addition, the three contributions* $-T_{\rm in}^2\boldsymbol{Q}_{\rm tr}\cdot\boldsymbol{\nabla}T_{\rm tr} - T_{\rm tr}^2\boldsymbol{Q}_{\rm in}\cdot\boldsymbol{\nabla}T_{\rm in}$, $-\boldsymbol{\Pi}:\boldsymbol{\nabla}\boldsymbol{v}$, *and* $\omega_{\rm in}(T_{\rm tr}-T_{\rm in})$ *are nonnegative.*

Proof. Using the expressions for the heat fluxes (2.14)(2.15), the viscous tensor (2.16), and the energy exchange rate (2.13), the entropy production $\mathfrak{v}$ is easily rewritten in the form

$$\begin{aligned}\mathfrak{v} = {} & \frac{\epsilon_d\bar{\lambda}_{\rm tr,tr}}{T_{\rm tr}^2}|\boldsymbol{\nabla}T_{\rm tr}|^2 + \frac{\epsilon_d\bar{\lambda}_{\rm tr,in}}{T_{\rm tr}^2}\boldsymbol{\nabla}T_{\rm tr}\cdot\boldsymbol{\nabla}T_{\rm in} + \frac{\epsilon_d\bar{\lambda}_{\rm in,tr}}{T_{\rm in}^2}\boldsymbol{\nabla}T_{\rm in}\cdot\boldsymbol{\nabla}T_{\rm tr} + \frac{\epsilon_d\bar{\lambda}_{\rm in,in}}{T_{\rm in}^2}|\boldsymbol{\nabla}T_{\rm in}|^2 \\ & + \frac{\epsilon_d\bar{\eta}}{T_{\rm tr}}\Big(\frac{1}{d}-\frac{1}{d'}\Big)(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2 + \frac{\epsilon_d\bar{\eta}}{2T_{\rm tr}}\big|\boldsymbol{\nabla}\boldsymbol{v}+\boldsymbol{\nabla}\boldsymbol{v}-\tfrac{2}{d}\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\boldsymbol{I}\big|^2 + \frac{1}{\epsilon}\frac{\rho c_{\rm in}}{\bar{\tau}_{\rm in}}\frac{(T_{\rm tr}-T_{\rm in})^2}{T_{\rm tr}T_{\rm in}},\end{aligned} \tag{2.22}$$

where for any matrix A we have denoted by $|A|$ the Frobenius norm with $|A|^2 = \sum_{1\le i,j\le d} A_{ij}^2$. The three last terms in (2.22) are then nonnegative as well as the sum of the four first thanks to the property of the matrix (2.19). □

2.5 Local equilibrium temperature

The local thermal equilibrium temperature is defined as the unique scalar T such that

$$e_{\rm tr}(T) + e_{\rm in}(T) = e_{\rm tr}(T_{\rm tr}) + e_{\rm in}(T_{\rm in}), \tag{2.23}$$

keeping in mind that $e_{\rm tr}(T)+e_{\rm in}(T)$ is an increasing function of T since $c_{\rm v,tr}$ and $c_{\rm in}$ are both positive. The temperature T is thus a $C^{\varkappa}$ function of $(T_{\rm tr}, T_{\rm in})$ and is the temperature that would be obtained locally at thermal equilibrium $T_{\rm tr} = T_{\rm in}$ assuming that the internal energy $e_{\rm tr}+e_{\rm in}$ is kept fixed. The function $e_{\rm in}(T_{\rm in})$ is generally nonlinear but we may write that $e_{\rm in}(T) - e_{\rm in}(T_{\rm in}) = (T - T_{\rm in})\widetilde{c}_{\rm in}$ where $\widetilde{c}_{\rm in} = \int_0^1 c_{\rm in}\big(T_{\rm in}+s(T-T_{\rm in})\big)\,{\rm d}s$. The relation $e_{\rm tr}(T_{\rm tr})-e_{\rm tr}(T) = e_{\rm in}(T)-e_{\rm in}(T_{\rm in})$ may thus be recast in the form $(T_{\rm tr}-T)c_{\rm v,tr} = (T-T_{\rm in})\widetilde{c}_{\rm in}$ or $(T_{\rm tr}-T)\widetilde{c}_{\rm v} = (T_{\rm tr}-T_{\rm in})\widetilde{c}_{\rm in}$ where $\widetilde{c}_{\rm v}(T_{\rm tr}, T_{\rm in}) = c_{\rm v,tr}+\widetilde{c}_{\rm in}(T_{\rm tr}, T_{\rm in})$.

On the other hand, after some algebra, the following equation is obtained for the temperature difference $T_{\rm tr} - T_{\rm in}$

$$\begin{aligned}\partial_t(T_{\rm tr}-T_{\rm in}) + \boldsymbol{v}\cdot\boldsymbol{\nabla}(T_{\rm tr}-T_{\rm in}) = {} & -\frac{p\boldsymbol{\nabla}\cdot\boldsymbol{v}}{\rho c_{\rm v,tr}} \\ & -\frac{1}{\rho c_{\rm v,tr}}\Big(\boldsymbol{\Pi}:\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\cdot\boldsymbol{Q}_{\rm tr} - \frac{c_{\rm v,tr}}{c_{\rm in}}\boldsymbol{\nabla}\cdot\boldsymbol{Q}_{\rm in}\Big) - \frac{c_{\rm v}}{c_{\rm v,tr}}\frac{T_{\rm tr}-T_{\rm in}}{\tau_{\rm in}},\end{aligned}$$

where we have defined $c_{\rm v}(T_{\rm in}) = c_{\rm v,tr} + c_{\rm in}(T_{\rm in})$. This is a relaxation equation which yields at leading order the approximation $T_{\rm tr} - T_{\rm in} \simeq -p\boldsymbol{\nabla}\cdot\boldsymbol{v}\tau_{\rm in}/(\rho c_{\rm v})$ so that the temperature difference is asymptotically proportional to the divergence of the velocity field. Defining the nonequilibrium volume viscosity by [3]

$$\kappa = \kappa(T_{\rm tr}, T_{\rm in}) = \frac{r\,\widetilde{c}_{\rm in}\,p\tau_{\rm in}}{c_{\rm v}\widetilde{c}_{\rm v}} = \frac{r\,\widetilde{c}_{\rm in}\,p^{\rm st}\tau_{\rm in}^{\rm st}}{c_{\rm v}\widetilde{c}_{\rm v}},$$

we thus obtain at leading order that $\rho r(T_{\rm tr}-T) \simeq -\kappa \boldsymbol{\nabla}\cdot\boldsymbol{v}$. In the fast relaxation limit with $\epsilon \to 0$ and $\tau_{\rm in} = \epsilon \bar{\tau}_{\rm in}$ we thus have $T_{\rm tr} - T = \mathcal{O}(\epsilon)$ as well as $T_{\rm tr} - T_{\rm in} = \mathcal{O}(\epsilon)$ and both temperatures $T_{\rm tr}$ and $T_{\rm in}$ converge towards the thermal equilibrium temperature T. We have consistently termed T the thermal equilibrium temperature and termed *equilibrium states* the states such that $T_{\rm tr} = T_{\rm in}$. After some algebra we further obtain that

$$\rho r(T_{\rm tr}-T) = -\kappa \boldsymbol{\nabla}\cdot\boldsymbol{v} - \frac{\kappa}{p}\Big(\boldsymbol{\Pi}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\cdot\boldsymbol{Q}_{\rm tr} - \frac{c_{\rm v,tr}}{c_{\rm in}}\boldsymbol{\nabla}\cdot\boldsymbol{Q}_{\rm in} + \rho\partial_t(T_{\rm tr}-T_{\rm in}) + \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}(T_{\rm tr}-T_{\rm in})\Big), \tag{2.24}$$

and since all conductivities and the shear viscosity are $\mathcal{O}(\epsilon_d)$ we deduce that

$$\rho r(T_{\rm tr}-T) = -\kappa_{\rm e}(T)\boldsymbol{\nabla}\cdot\boldsymbol{v} + \mathcal{O}\big(\epsilon(\epsilon+\epsilon_d)\big), \tag{2.25}$$

where $\kappa_{\rm e}(T) = \kappa(T,T)$. In the momentum equation, the pressure term $\rho r T_{\rm tr}$ may then be written $\rho r T - \kappa_{\rm e}\boldsymbol{\nabla}\cdot\boldsymbol{v}$ and the pressure tensor $\rho r T_{\rm tr}\boldsymbol{I} + \boldsymbol{\Pi}$ is asymptotically in the form

$$\rho r T_{\rm tr}\boldsymbol{I} + \boldsymbol{\Pi} = \rho r T\boldsymbol{I} - \kappa_{\rm e}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{I} - \eta_{\rm e}\big(\boldsymbol{\nabla}\boldsymbol{v} + (\boldsymbol{\nabla}\boldsymbol{v})^t - \tfrac{2}{d'}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{I}\big) + \mathcal{O}\big(\epsilon(\epsilon+\epsilon_d)\big),$$

where $\eta_{\rm e}(T) = \eta(T,T)$. This is in agreement with classical one-temperature models where the pressure $\rho r T$ is evaluated at the thermal equilibrium temperature T and the viscous tensor includes a volume viscosity term $-\kappa_{\rm e}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{I}$.

Such a derivation may be found in many physics papers and books either in a molecular framework or in a macroscopic fluid framework usually around the equilibrium state [27, 46, 6, 16, 34, 37, 38, 3, 4, 5]. Numerical simulations using the Boltzmann equation have consistently established that the limit one-temperature model is an accurate description of the two temperature fluid when the relaxation time is small [3]. One of the goals of this paper is to justify rigorously the above physical traditional analysis.

2.6 Quasilinear form

Letting $n = d+3$, the conservative variable $\mathsf{u} \in \mathbb{R}^n$ associated with equations (2.1)–(2.4) is found to be

$$\mathsf{u} = \big(\rho, \rho\boldsymbol{v}, \mathcal{E}_{\rm in}, \mathcal{E}_{\rm tr} + \mathcal{E}_{\rm in} + \tfrac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v}\big)^t, \tag{2.26}$$

and the natural variable $\mathsf{z} \in \mathbb{R}^n$ is defined by

$$\mathsf{z} = \big(\rho, \boldsymbol{v}, T_{\rm in}, T_{\rm tr}\big)^t. \tag{2.27}$$

For convenience, the velocity components of vectors in $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^2$ are generally written as vectors of $\mathbb{R}^d$. We introduce the corresponding open sets $\mathcal{O}_{\mathsf{u}}$ and $\mathcal{O}_{\mathsf{z}}$ of $\mathbb{R}^n$ given by

$$\mathcal{O}_{\mathsf{u}} = \big\{\mathsf{u} = (\mathsf{u}_\rho, \mathsf{u}_{\boldsymbol{v}}, \mathsf{u}_{\rm in}, \mathsf{u}_{\rm tl})^t \in \mathbb{R}^n;\ \mathsf{u}_\rho > 0,\ \mathsf{u}_{\rm in} > \mathsf{u}_\rho e^0_{\rm in},\ \mathsf{u}_{\rm tl} > f(\mathsf{u}_\rho, \mathsf{u}_{\boldsymbol{v}}, \mathsf{u}_{\rm in})\big\}, \tag{2.28}$$

where $f : (0,\infty)\times\mathbb{R}^d\times(0,\infty) \to \mathbb{R}$ is given by $f(\mathsf{u}_\rho, \mathsf{u}_{\boldsymbol{v}}, \mathsf{u}_{\rm in}) = \mathsf{u}_{\rm in} + \frac{1}{2}\mathsf{u}_{\boldsymbol{v}}\cdot\mathsf{u}_{\boldsymbol{v}}/\mathsf{u}_\rho$ and

$$\mathcal{O}_{\mathsf{z}} = (0,\infty)\times\mathbb{R}^d\times(0,\infty)^2. \tag{2.29}$$

Proposition 2.3. *Assuming that* (T_1) *holds, the map* $\mathsf{z} \longmapsto \mathsf{u}$ *is a* $C^\varkappa$ *diffeomorphism from the open set* $\mathcal{O}_{\mathsf{z}}$ *onto the open set* $\mathcal{O}_{\mathsf{u}}$ *and the open set* $\mathcal{O}_{\mathsf{u}}$ *is convex.*

Proof. We first establish that the map $\mathsf{z} \longmapsto \mathsf{u}$ is one to one. Assuming that $\mathsf{u}(\mathsf{z}^\sharp) = \mathsf{u}(\mathsf{z}^\flat)$ for $\mathsf{z}^\sharp, \mathsf{z}^\flat \in \mathcal{O}_{\mathsf{z}}$, the corresponding mass densities then coincide $\rho^\sharp = \rho^\flat$, as well as the velocities $\boldsymbol{v}^\sharp = \boldsymbol{v}^\flat$ with straightforward notation. Furthermore, the energies $\mathcal{E}^\sharp_{\rm in} = \mathcal{E}^\flat_{\rm in}$ also coincide so that $e^\sharp_{\rm in} = e^\flat_{\rm in}$ and we may use the positivity of the specific heat $c_{\rm in}$ to deduce that $T^\sharp_{\rm in} = T^\flat_{\rm in}$. Finally, we also have $\mathcal{E}^\sharp_{\rm tr} = \mathcal{E}^\flat_{\rm tr}$ so that $T^\sharp_{\rm tr} = T^\flat_{\rm tr}$ and the map is one to one. Moreover, the map $\mathsf{z} \longmapsto \mathsf{u}$ is $C^\varkappa$ over the open set $\mathcal{O}_{\mathsf{z}}$ since $e_{\rm in}$ is a $C^\varkappa$ function of $T_{\rm in}$.

On the other hand, the Jacobian matrix $\partial_{\mathsf{z}}\mathsf{u}$ has a triangular structure

$$\partial_{\mathsf{z}}\mathsf{u} = \begin{bmatrix} 1 & 0_{1,d} & 0 & 0 \\ \boldsymbol{v} & \rho\boldsymbol{I} & 0_{d,1} & 0_{d,1} \\ e_{\mathrm{in}} & 0_{1,d} & \rho c_{\mathrm{in}} & 0 \\ e_{\mathrm{tl}} & \rho\boldsymbol{v}^t & \rho c_{\mathrm{in}} & \rho c_{\mathrm{v,tr}} \end{bmatrix},$$

where $0_{i,j}$ denotes a zero matrix with i lines and j columns and e_{tl} the total energy per unit mass $e_{\mathrm{tl}} = e + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} = e_{\mathrm{tr}} + e_{\mathrm{in}} + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}$. Since $c_{\mathrm{in}} > 0$, $c_{\mathrm{v,tr}} > 0$, and $\rho > 0$ we deduce that $\partial_{\mathsf{z}}\mathsf{u}$ is invertible. From the inverse function theorem, the map $\mathsf{z} \longmapsto \mathsf{u}$ is a local $C^{\varkappa}$ diffeomorphism and its image is an open set. Since $\mathsf{z} \longmapsto \mathsf{u}$ is one to one it is thus a global $C^{\varkappa}$ diffeomorphism. From the construction of u and (T_1), it is easily established that the range of $\mathsf{z} \to \mathsf{u}$ is the open set $\mathcal{O}_{\mathsf{u}}$ defined by (2.28). Indeed, denoting by $\mathsf{u} = (\mathsf{u}_\rho, \mathsf{u}_{\boldsymbol{v}}, \mathsf{u}_{\mathrm{in}}, \mathsf{u}_{\mathrm{tl}})^t$ the components of the conservative variable u, the condition $\mathsf{u}_{\mathrm{in}} > \mathsf{u}_\rho e^0_{\mathrm{in}}$ is equivalent to $T_{\mathrm{in}} > 0$ and $\mathsf{u}_{\mathrm{tl}} > f(\mathsf{u}_\rho, \mathsf{u}_{\boldsymbol{v}}, \mathsf{u}_{\mathrm{in}})$ is equivalent to $T_{\mathrm{tr}} > 0$.

In order to establish that $\mathcal{O}_{\mathsf{u}}$ is convex, it is sufficient to establish that f is convex over $\mathcal{O}_{\mathsf{z}}$. Denoting by $\mathsf{u} = (\mathsf{u}_\rho, \mathsf{u}_{\boldsymbol{v}}, \mathsf{u}_{\mathrm{in}}, \mathsf{u}_{\mathrm{tl}})^t$ the components of the conservative variable we obtain that $\partial_{\mathsf{u}_\rho} f = -\frac{\mathsf{u}_{\boldsymbol{v}}\cdot\mathsf{u}_{\boldsymbol{v}}}{\mathsf{u}_\rho^2}$, $\partial_{\mathsf{u}_{\boldsymbol{v}}} f = 2\frac{\mathsf{u}_{\boldsymbol{v}}^t}{\mathsf{u}_\rho}$, $\partial_{\mathsf{u}_{\mathrm{in}}} f = 1$. Similarly we have $\partial^2_{\mathsf{u}_\rho} f = 2\frac{\mathsf{u}_{\boldsymbol{v}}\cdot\mathsf{u}_{\boldsymbol{v}}}{\mathsf{u}_\rho^3}$, $\partial^2_{\mathsf{u}_\rho\,\mathsf{u}_{\boldsymbol{v}}} f = -2\frac{\mathsf{u}_{\boldsymbol{v}}^t}{\mathsf{u}_\rho^2}$, $\partial^2_{\mathsf{u}_{\boldsymbol{v}}} f = 2\frac{1}{\mathsf{u}_\rho^2}\boldsymbol{I}$ and all second derivatives involving u_{in} vanish, so that for any $\mathsf{x} \in \mathbb{R}^{d+2}$

$$\langle \partial^2_{\mathsf{u}} f\, \mathsf{x}, \mathsf{x}\rangle = \frac{2}{\mathsf{u}_\rho^3}(\mathsf{u}_\rho \mathsf{x}_{\boldsymbol{v}} - \mathsf{u}_{\boldsymbol{v}}\mathsf{x}_\rho)\cdot(\mathsf{u}_\rho \mathsf{x}_{\boldsymbol{v}} - \mathsf{u}_{\boldsymbol{v}}\mathsf{x}_\rho).$$

Hence $\partial^2_{\mathsf{u}} f$ is positive semi-definite so that f is convex and the proof is complete. □

The equations modeling fluids out of thermodynamic equilibrium may then be written in the compact form

$$\partial_t \mathsf{u} + \sum_{i\in\mathcal{D}} \partial_i \mathsf{F}_i + \epsilon_d \sum_{i\in\mathcal{D}} \partial_i \mathsf{F}_i^{\mathrm{diss}} - \frac{1}{\epsilon}\Omega = 0, \tag{2.30}$$

where F_i is the convective flux in the ith direction, ϵ_d the Knudsen number, $\mathsf{F}_i^{\mathrm{diss}}$ the rescaled dissipative flux in the ith direction, ϵ the relaxation parameter, Ω the rescaled source term, and $\mathcal{D} = \{1, \dots, d\}$ the indexing set of spatial dimensions.

From the governing equations (2.1)–(2.4) the convective flux F_i in the ith direction is given by

$$\mathsf{F}_i = \big(\rho v_i,\, \rho\boldsymbol{v}v_i + p\boldsymbol{e}_i, \mathcal{E}_{\mathrm{in}} v_i, (\mathcal{E}_{\mathrm{tr}} + \mathcal{E}_{\mathrm{in}} + p + \tfrac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v})v_i\big)^t, \tag{2.31}$$

where $\boldsymbol{e}_i$ denotes the basis vectors of $\mathbb{R}^d$. Similarly, the dissipative flux $\epsilon_d \mathsf{F}_i^{\mathrm{diss}}$ is given by

$$\epsilon_d \mathsf{F}_i^{\mathrm{diss}} = \big(0,\, \boldsymbol{\Pi}_i,\, Q_{\mathrm{in},i},\, Q_{\mathrm{tr},i} + Q_{\mathrm{in},i} + \boldsymbol{\Pi}_i\cdot\boldsymbol{v}\big)^t, \tag{2.32}$$

where $\boldsymbol{\Pi}_i$ denotes the vector $\boldsymbol{\Pi}_i = (\Pi_{1i}, \dots, \Pi_{di})^t$, and the source term is given by

$$\frac{1}{\epsilon}\Omega = \big(0, \mathbf{0}, \omega_{\mathrm{in}}, 0\big)^t. \tag{2.33}$$

From the expressions of the viscous tensor and of the heat fluxes we deduce that the dissipative fluxes $\mathsf{F}_i^{\mathrm{diss}}$ may be written in the form $\mathsf{F}_i^{\mathrm{diss}} = -\sum_{j\in\mathcal{D}} \widehat{\mathsf{B}}_{ij}(\mathsf{z})\partial_j \mathsf{z}$ where $\widehat{\mathsf{B}}_{ij}$ denotes the dissipation matrix relating the rescaled flux $\mathsf{F}_i^{\mathrm{diss}}$ in the ith direction with the gradient of the natural variable $\partial_j \mathsf{z}$ in the jth direction. These matrices $\widehat{\mathsf{B}}_{ij}$ are square matrices of size $n = d+3$ that are directly written in terms of rescaled transport coefficients and thermodynamic properties. Thanks to Proposition 2.3, we may then write that $\mathsf{F}_i^{\mathrm{diss}} = -\sum_{j\in\mathcal{D}} \mathsf{B}_{ij}(\mathsf{u})\partial_j \mathsf{u}$ where the dissipation matrix B_{ij} is defined as $\mathsf{B}_{ij} = \widehat{\mathsf{B}}_{ij}\partial_{\mathsf{u}}\mathsf{z}$. Further introducing the Jacobian matrices of the convective fluxes $\mathsf{A}_i = \partial_{\mathsf{u}}\mathsf{F}_i$ the governing equations are finally rewritten in the form of a quasilinear system in terms of the conservative variable u whose structure is discussed in Section 3.

3 Symmetrization and normal form

We discuss symmetrization with entropic variables and normal variables for abstract systems with small second order terms and stiff sources as well as source terms in quasilinear form [51]. We then explicitly evaluate the natural entropic symmetrized form and a normal form for the system of partial differential equations modeling fluids out of thermodynamic equilibrium.

3.1 Entropic variables

We consider an abstract second order quasilinear system of conservation laws in the general form

$$\partial_t \mathsf{u} + \sum_{i\in\mathcal{D}} \mathsf{A}_i(\mathsf{u})\partial_i \mathsf{u} - \epsilon_d \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}_{ij}(\mathsf{u})\partial_j \mathsf{u}\big) - \frac{1}{\epsilon}\Omega(\mathsf{u}) = 0, \tag{3.1}$$

where $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$, $\mathcal{O}_{\mathsf{u}}$ is an open convex set of $\mathbb{R}^n$, $n \geq 1$, and ϵ_d and ϵ are positive parameters. The system coefficients are such that $\mathsf{A}_i = \partial_{\mathsf{u}}\mathsf{F}_i$ where F_i, $i \in \mathcal{D}$, are fluxes, and we assume that the fluxes F_i, $i \in \mathcal{D}$, the dissipation matrices B_{ij}, $i,j \in \mathcal{D}$, and the source term Ω, are $C^{\varkappa}$ over $\mathcal{O}_{\mathsf{u}}$ where $\varkappa \geq 3$.

We use the definition of a mathematical entropy for dissipative systems of conservation laws with source terms presented in [22, 23] and simplified to the situation where the set $\mathcal{O}_{\mathsf{u}}$ is convex. Properties $(\mathsf{E}_1)(\mathsf{E}_2)$ have been adapted from [25, 17], properties $(\mathsf{E}_3)(\mathsf{E}_4)$ from [30, 44, 42, 31, 32] and properties (E_5)-(E_7) from [7, 33] and we denote by Σ^{d-1} the sphere in d dimension.

Definition 3.1. *Consider a $C^{\varkappa}$ function $\mathsf{u} \to \sigma(\mathsf{u})$ defined over the open convex domain $\mathcal{O}_{\mathsf{u}}$. The function σ is said to be an entropy function for the system* (3.1) *if the following properties hold.*

(E_1) *The Hessian matrix $\partial^2_{\mathsf{u}}\sigma(\mathsf{u}) = \partial_{\mathsf{u}}(\partial_{\mathsf{u}}\sigma)^t(\mathsf{u})$ is positive definite over $\mathcal{O}_{\mathsf{u}}$.*

(E_2) *There exist real-valued $C^{\varkappa}$ functions $\mathsf{u} \to \mathsf{q}_i(\mathsf{u})$ such that $\partial_{\mathsf{u}}\sigma(\mathsf{u})\,\mathsf{A}_i(\mathsf{u}) = \partial_{\mathsf{u}}\mathsf{q}_i(\mathsf{u})$ for $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$ and $i \in \mathcal{D}$.*

(E_3) *We have $\Big(\mathsf{B}_{ij}(\mathsf{u})\big(\partial^2_{\mathsf{u}}\sigma(\mathsf{u})\big)^{-1}\Big)^t = \mathsf{B}_{ji}(\mathsf{u})\big(\partial^2_{\mathsf{u}}\sigma(\mathsf{u})\big)^{-1}$ for $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$ and $i,j \in \mathcal{D}$.*

(E_4) *The matrix $\widetilde{\mathsf{B}}(\mathsf{u},\boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}} \mathsf{B}_{ij}(\mathsf{u})\big(\partial^2_{\mathsf{u}}\sigma(\mathsf{u})\big)^{-1}\xi_i\xi_j$ is positive semi-definite for $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$.*

(E_5) *There exists a fixed vector space $\mathcal{E} \subset \mathbb{R}^n$ such that $\Omega(\mathsf{u}) \in \mathcal{E}^{\perp}$ for $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$ and $\Omega(\mathsf{u}) = 0$ if and only if $\big(\partial_{\mathsf{u}}\sigma(\mathsf{u})\big)^t \in \mathcal{E}$ and if and only if $\partial_{\mathsf{u}}\sigma(\mathsf{u})\ \Omega(\mathsf{u}) = 0$.*

(E_6) *If $\Omega(\mathsf{u}) = 0$, then the matrix $\partial_{\mathsf{u}}\Omega(\mathsf{u})\big(\partial^2_{\mathsf{u}}\sigma(\mathsf{u})\big)^{-1}$ is symmetric with its nullspace given by $N\Big(\partial_{\mathsf{u}}\Omega(\mathsf{u})\big(\partial^2_{\mathsf{u}}\sigma(\mathsf{u})\big)^{-1}\Big) = \mathcal{E}$.*

(E_7) *We have $\partial_{\mathsf{u}}\sigma(\mathsf{u})\ \Omega(\mathsf{u}) \leq 0$ for $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$.*

Existence of an entropy is closely associated with symmetrization properties [25, 17, 30, 44, 42, 31, 32, 7, 33, 22, 23]. We do not encounter here the difficulty associated with nonideal fluids where only *local* symmetrization are feasible and where $\mathcal{O}_{\mathsf{u}}$ may not be convex [21, 22]. Note also that more general source terms with no symmetry properties at equilibrium have been considered by Chen, Levermore and Liu [7] and Yong [50]. The following definition of symmetrizability is associated with definition 3.1 of mathematical entropy but weaker definition may also be used [50, 47].

Definition 3.2. *Consider a $C^{\varkappa-1}$ diffeomorphism $\mathsf{u} \to \mathsf{v}$ from $\mathcal{O}_{\mathsf{u}}$ onto an open domain $\mathcal{O}_{\mathsf{v}}$ and the system in the v variable*

$$\widetilde{\mathsf{A}}_0(\mathsf{v})\partial_t \mathsf{v} + \sum_{i\in\mathcal{D}} \widetilde{\mathsf{A}}_i(\mathsf{v})\partial_i \mathsf{v} - \epsilon_d \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}_{ij}(\mathsf{v})\partial_j \mathsf{v}\big) - \frac{1}{\epsilon}\widetilde{\Omega}(\mathsf{v}) = 0, \tag{3.2}$$

where $\widetilde{\mathsf{A}}_0 = \partial_{\mathsf{v}}\mathsf{u}$, $\widetilde{\mathsf{A}}_i = \mathsf{A}_i\partial_{\mathsf{v}}\mathsf{u} = \partial_{\mathsf{v}}\mathsf{F}_i$, $\widetilde{\mathsf{B}}_{ij} = \mathsf{B}_{ij}\partial_{\mathsf{v}}\mathsf{u}$, and $\widetilde{\Omega} = \Omega$, have at least regularity $\varkappa - 2$. The system is said of the symmetric form if properties (S_1)-(S_7) *hold.*

(S_1) *The matrix* $\widetilde{\mathsf{A}}_0(\mathsf{v})$ *is symmetric positive definite for* $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$.

(S_2) *The matrices* $\widetilde{\mathsf{A}}_i(\mathsf{v})$, $i \in C$, *are symmetric for* $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$.

(S_3) *We have* $\widetilde{\mathsf{B}}_{ij}^t(\mathsf{v}) = \widetilde{\mathsf{B}}_{ji}(\mathsf{v})$ *for* $i, j \in C$ *and* $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$.

(S_4) *The matrix* $\widetilde{\mathsf{B}}(\mathsf{v}, \boldsymbol{\xi}) = \sum_{i,j \in \mathcal{D}} \widetilde{\mathsf{B}}_{ij}(\mathsf{v})\xi_i\xi_j$ *is positive semi-definite for* $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$ *and* $\boldsymbol{\xi} \in \Sigma^{d-1}$.

(S_5) *There exists a fixed vector space* $\mathcal{E} \subset \mathbb{R}^n$ *such that* $\widetilde{\Omega}(\mathsf{v}) \in \mathcal{E}^\perp$ *for* $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$ *and* $\widetilde{\Omega}(\mathsf{v}) = 0$ *if and only if* $\mathsf{v} \in \mathcal{E}$ *and if and only if* $\langle \mathsf{v}, \widetilde{\Omega}(\mathsf{v}) \rangle = 0$.

(S_6) *If* $\widetilde{\Omega}(\mathsf{v}) = 0$, *then* $\partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v})$ *is symmetric and* $N\big(\partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v})\big) = \mathcal{E}$.

(S_7) *We have* $\langle \mathsf{v}, \widetilde{\Omega}(\mathsf{v}) \rangle \leq 0$ *for* $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$.

The manifold $\mathcal{E}$ is naturally termed the equilibrium manifold or the slow manifold, since $\widetilde{\Omega}(\mathsf{v}) = 0$ when $\mathsf{v} \in \mathcal{E}$, and $\mathcal{E}^\perp$ is naturally termed the fast manifold since there is a fast variation of v with large values of $\widetilde{\Omega}$ along the directions of $\mathcal{E}^\perp$. The equivalence between symmetrization (S_1)-(S_7) and entropy (E_1)-(E_7) for hyperbolic-parabolic systems of conservation laws is obtained with $\mathsf{v} = (\partial_{\mathsf{u}}\sigma)^t$ [22].

Theorem 3.3. *Assume that the system* (3.1) *admits a* $C^\varkappa$ *entropy function* σ *defined over an open convex domain* $\mathcal{O}_{\mathsf{u}}$. *Then the system can be symmetrized with the entropic variable* $\mathsf{v} = (\partial_{\mathsf{u}}\sigma)^t$. *Conversely, assume that the system can be symmetrized with the* $C^{\varkappa-1}$ *diffeomorphism* $\mathsf{u} \to \mathsf{v}$. *Then there exists a* $C^\varkappa$ *entropy over the open convex set* $\mathcal{O}_{\mathsf{u}}$ *such that* $\mathsf{v} = (\partial_{\mathsf{u}}\sigma)^t$.

We further investigate quasilinear symmetrized source terms $\widetilde{\Omega}$ which naturally arise in various areas of mathematical physics [51] and are typical in the context of relaxation phenomena.

Definition 3.4. *The symmetrized source terms* $\widetilde{\Omega}$ *is said to be in quasilinear form [51] if there exists a* $C^{\varkappa-1}$ *map* $\mathsf{v} \to \widetilde{\mathsf{L}}(\mathsf{v})$ *where* $\widetilde{\mathsf{L}}$ *is a square matrix of order* n *such that*

$$\widetilde{\Omega}(\mathsf{v}) = -\widetilde{\mathsf{L}}(\mathsf{v})\mathsf{v}, \tag{3.3}$$

and which satisfies the following properties (L_1)-(L_2).

(L_1) *The matrix* $\widetilde{\mathsf{L}}(\mathsf{v})$ *is symmetric positive semi-definite for* $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$.

(L_2) *There exists a fixed vector space* $\mathcal{E} \subset \mathbb{R}^n$ *such that* $N\big(\widetilde{\mathsf{L}}(\mathsf{v})\big) = \mathcal{E}$ *for* $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$.

The structural properties of the source term (S_5)-(S_7) associated with the symmetrized form are then automatically satisfied as established in the following lemma.

Lemma 3.5. *Assume that the source term* $\widetilde{\Omega}$ *is in quasilinear form as in Definition 3.4. Then properties* (S_5)-(S_7) *are automatically satisfied with the same equilibrium manifold* $\mathcal{E}$.

Proof. From the symmetry of $\widetilde{\mathsf{L}}$ we deduce that $\widetilde{\Omega}(\mathsf{v}) \in \mathcal{E}^\perp$ for $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$ because $N(\widetilde{\mathsf{L}}) = \mathcal{E}$. Since $\widetilde{\mathsf{L}}$ is positive semi-definite, we also obtain that $\widetilde{\Omega} = 0$ if and only if $\mathsf{v} \in \mathcal{E}$ and if and only if $\langle \mathsf{v}, \Omega(\mathsf{v}) \rangle = -\langle \mathsf{v}, \widetilde{\mathsf{L}}(\mathsf{v})\mathsf{v} \rangle = 0$.

Moreover, for any $x \in \mathcal{E}$ we have $\widetilde{\mathsf{L}}(\mathsf{v})x = 0$ so that $\partial_{\mathsf{v}}\widetilde{\mathsf{L}}(\mathsf{v})x = 0$. Therefore, for any $\mathsf{v} \in \mathcal{E}$, we obtain that $\partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v}) = -\partial_{\mathsf{v}}\widetilde{\mathsf{L}}(\mathsf{v})\mathsf{v} - \widetilde{\mathsf{L}}(\mathsf{v}) = -\widetilde{\mathsf{L}}(\mathsf{v})$. This shows that at equilibrium $\partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v}) = -\widetilde{\mathsf{L}}(\mathsf{v})$ so that $\partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v})$ is symmetric and $N(\partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v})) = \mathcal{E}$. Finally, it is straightforward to check that $\langle \mathsf{v}, \Omega(\mathsf{v}) \rangle = -\langle \mathsf{v}, \widetilde{\mathsf{L}}(\mathsf{v})\mathsf{v} \rangle \leq 0$ and the proof is complete. $\square$

3.2 Normal variables

In order to split the variables between hyperbolic and parabolic variables, we further have to put the system into a normal form, that is, in the form of a symmetric hyperbolic-parabolic composite system [30, 32, 19].

Definition 3.6. *Consider a symmetrized system as in Definition 3.2 and let* $\mathsf{v} \to \mathsf{w}$ *be a* $C^{\varkappa-1}$ *diffeomorphism from the open set* $\mathcal{O}_{\mathsf{v}}$ *onto an open set* $\mathcal{O}_{\mathsf{w}}$*. Letting* $\mathsf{v} = \mathsf{v}(\mathsf{w})$ *in the symmetrized system (3.2) and multiplying on the left side by* $(\partial_{\mathsf{w}}\mathsf{v})^t$ *we obtain a new system in the variable* w

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\mathsf{w} \ +\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\mathsf{w} - \epsilon_d \sum_{i,j\in\mathcal{D}}\partial_i\big(\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_j\mathsf{w}\big) - \frac{1}{\epsilon}\overline{\Omega}(\mathsf{w}) = \epsilon_d\overline{\mathsf{b}}(\mathsf{w},\partial_{\mathsf{x}}\mathsf{w}), \tag{3.4}$$

where $\overline{\mathsf{A}}_0 = (\partial_{\mathsf{w}}\mathsf{v})^t\, \widetilde{\mathsf{A}}_0\, (\partial_{\mathsf{w}}\mathsf{v})$, $\overline{\mathsf{B}}_{ij} = (\partial_{\mathsf{w}}\mathsf{v})^t\, \widetilde{\mathsf{B}}_{ij}\, (\partial_{\mathsf{w}}\mathsf{v})$, $\overline{\mathsf{A}}_i = (\partial_{\mathsf{w}}\mathsf{v})^t\, \widetilde{\mathsf{A}}_i\, (\partial_{\mathsf{w}}\mathsf{v})$, $\overline{\Omega} = (\partial_{\mathsf{w}}\mathsf{v})^t\widetilde{\Omega}$*, have at least regularity* $\varkappa-2$*, and where* $\overline{\mathsf{b}}$ *is quadratic in the gradients* $\overline{\mathsf{b}} = -\sum_{i,j\in\mathcal{D}}\partial_i(\partial_{\mathsf{w}}\mathsf{v})^t\,\widetilde{\mathsf{B}}_{ij}\,(\partial_{\mathsf{w}}\mathsf{v})\partial_j\mathsf{w}$*. This system satisfies in particular properties* $(\overline{\mathsf{S}}_1)$-$(\overline{\mathsf{S}}_4)$*, that is, properties* (S_1)-(S_4) *rewritten in terms of overbar matrices. This system (3.4) is said to be of the normal form if there exists a partition of* $\{1,\dots,n\}$ *into* $\mathrm{I} = \{1,\dots,n_{\mathrm{I}}\}$ *and* $\mathrm{II} = \{n_{\mathrm{I}}+1,\dots,n_{\mathrm{I}}+n_{\mathrm{II}}\}$ *with* $n = n_{\mathrm{I}}+n_{\mathrm{II}}$ *such that the following properties hold.*

(N_1) *The matrices* $\overline{\mathsf{A}}_0$ *and* $\overline{\mathsf{B}}_{ij}$ *have the block structure*

$$\overline{\mathsf{A}}_0 = \begin{bmatrix} \overline{\mathsf{A}}_0^{\mathrm{I,I}} & 0_{n_{\mathrm{I}},n_{\mathrm{II}}} \\ 0_{n_{\mathrm{II}},n_{\mathrm{I}}} & \overline{\mathsf{A}}_0^{\mathrm{II,II}} \end{bmatrix}, \qquad \overline{\mathsf{B}}_{ij} = \begin{bmatrix} 0_{n_{\mathrm{I}},n_{\mathrm{I}}} & 0_{n_{\mathrm{I}},n_{\mathrm{II}}} \\ 0_{n_{\mathrm{II}},n_{\mathrm{I}}} & \overline{\mathsf{B}}_{ij}^{\mathrm{II,II}} \end{bmatrix}.$$

(N_2) *The matrix* $\overline{\mathsf{B}}^{\mathrm{II,II}}(\mathsf{w},\boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}^{\mathrm{II,II}}(\mathsf{w})\xi_i\xi_j$ *is positive definite for* $\mathsf{w}\in\mathcal{O}_{\mathsf{w}}$ *and* $\boldsymbol{\xi}\in\Sigma^{d-1}$*.*

(N_3) *We have* $\overline{\mathsf{b}}(\mathsf{w},\partial_{\mathsf{x}}\mathsf{w}) = \big(\overline{\mathsf{b}}_{\mathrm{I}}(\mathsf{w},\partial_{\mathsf{x}}\mathsf{w}_{\mathrm{II}}), \overline{\mathsf{b}}_{\mathrm{II}}(\mathsf{w},\partial_{\mathsf{x}}\mathsf{w})\big)^t$*.*

We have used here the vector and matrix block structure induced by the partitioning of $\mathbb{R}^n$ *into* $\mathbb{R}^n = \mathbb{R}^{n_{\mathrm{I}}}\times\mathbb{R}^{n_{\mathrm{II}}}$ *so that we have* $\mathsf{w} = (\mathsf{w}_{\mathrm{I}},\mathsf{w}_{\mathrm{II}})^t$ *for instance.*

The quadratic residual may also be written in the more elegant form

$$\overline{\mathsf{b}} = \sum_{i,j\in\mathcal{D}}\overline{\mathrm{M}}_{ij}(\mathsf{w})\,\partial_i\mathsf{w}\,\partial_j\mathsf{w}, \tag{3.5}$$

where $\overline{\mathrm{M}}_{ij}(\mathsf{w})$ are third order tensors that are functions of $\mathsf{w}\in\mathcal{O}_{\mathsf{w}}$. From the regularity assumptions of the original system (3.1), the coefficients of both symmetrized systems (3.2) and (3.4) have at least regularity $\varkappa-2$ and the coefficients $\overline{\mathrm{M}}_{ij}$, $i,j\in\mathcal{D}$, of $\overline{\mathsf{b}}$ have at least regularity $\varkappa-3$. A sufficient condition for system (3.2) to be recast into a normal form is that the nullspace naturally associated with dissipation matrices $\widetilde{\mathsf{B}}$ is a fixed subspace of $\mathbb{R}^n$. This is Condition (N) introduced by Kawashima and Shizuta [32] which has been strengthened in [19] :

(N) *The nullspace* $N(\widetilde{\mathsf{B}})$ *of the matrix* $\widetilde{\mathsf{B}}(\mathsf{v},\boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}}\widetilde{\mathsf{B}}_{ij}(\mathsf{v})\xi_i\xi_j$ *does not depend on* $\mathsf{v}\in\mathcal{O}_{\mathsf{v}}$ *and* $\boldsymbol{\xi}\in\Sigma^{d-1}$ *and* $\widetilde{\mathsf{B}}_{ij}(\mathsf{v})N(\widetilde{\mathsf{B}}) = 0$*, for* $i,j\in\mathcal{D}$*.*

Letting $n_{\mathrm{I}} = \dim\big(N(\widetilde{\mathsf{B}})\big)$ and $n_{\mathrm{II}} = n - n_{\mathrm{I}}$ we denote by P an arbitrary constant nonsingular matrix of dimension n such that its first n_{I} columns span the nullspace $N(\widetilde{\mathsf{B}})$. In order to characterize more easily normal forms for symmetric systems of conservation laws satisfying (N) we may introduce the auxiliary variables [19, 18] $\mathsf{u}' = \mathsf{P}^t\mathsf{u}$ and $\mathsf{v}' = \mathsf{P}^{-1}\mathsf{v}$. The dissipation matrices corresponding to these auxiliary variables have nonzero coefficients only in the lower right block of size $n_{\mathrm{II}} = n - n_{\mathrm{I}}$. Normal symmetric forms are then equivalently—and more easily—obtained from the v' symmetric equation [19, 18].

Theorem 3.7. *Consider a system of conservation laws (3.2) that is symmetric in the sense of Definition 3.2 and assume that the nullspace invariance property* (N) *is satisfied. Denoting by* $\mathsf{u}' = \mathsf{P}^t\mathsf{u}$ *and* $\mathsf{v}' = \mathsf{P}^{-1}\mathsf{v}$*, the auxiliary variable, any normal form of the system (3.2) is given by a change of variable in the form* $\mathsf{w} = \big(\mathcal{F}_{\mathrm{I}}(\mathsf{u}'_{\mathrm{I}}), \mathcal{F}_{\mathrm{II}}(\mathsf{v}'_{\mathrm{II}})\big)^t$ *where* $\mathcal{F}_{\mathrm{I}}$ *and* $\mathcal{F}_{\mathrm{II}}$ *are two diffeomorphisms of* $\mathbb{R}^{n_{\mathrm{I}}}$ *and* $\mathbb{R}^{n_{\mathrm{II}}}$*, respectively, and we have*

$$\overline{\mathsf{b}} = \Big(0,\, \overline{\mathsf{b}}_{\mathrm{II}}(\mathsf{w}, \partial_{\mathsf{x}}\mathsf{w}_{\mathrm{II}})\Big)^t = \Big(0,\, \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}_{ij}^{\mathrm{II},\mathrm{II},\mathrm{II}}(\mathsf{w})\, \partial_i \mathsf{w}_{\mathrm{II}}\, \partial_j \mathsf{w}_{\mathrm{II}}\Big)^t, \tag{3.6}$$

where $\overline{\mathrm{M}}_{ij}^{\mathrm{II},\mathrm{II},\mathrm{II}}(\mathsf{w})$ *are third order tensors depending on* w *with regularity at least* $\varkappa - 3$*. Finally, when* $\mathcal{F}_{\mathrm{II}}$ *is linear, the quadratic residual* $\overline{\mathsf{b}}$ *is zero.*

The main interest of normal forms is that the resulting subsystem of partial differential equations governing the variable w_{I} is symmetric hyperbolic [30, 40] whereas the subsystem governing w_{II} is symmetric strongly parabolic [30, 32]. Note incidentally that for second order systems of partial differential equations in such a symmetric form involving a 'mass matrix' $\overline{\mathsf{A}}_0$, strong parabolicity is equivalent to Petrovsky parabolicity [22].

We investigate in the following the situation where the general structure of the symmetrized source term $\widetilde{\Omega}$ is transfered to the source term $\overline{\Omega}$ of the normal variable. We consider a given normal change of variable $\mathsf{v} \to \mathsf{w}$ and we naturally define the image $\overline{\mathcal{E}}$ of the equilibrium manifold $\mathcal{E}$ by the equivalent properties

$$\overline{\mathcal{E}} = (\partial_{\mathsf{w}}\mathsf{v})^{-1}\mathcal{E}, \qquad \overline{\mathcal{E}}^{\perp} = (\partial_{\mathsf{w}}\mathsf{v})^t \mathcal{E}^{\perp}. \tag{3.7}$$

Definition 3.8. *Denoting by* $\tilde{\pi}$ *the orthogonal projector onto the fast manifold* $\mathcal{E}^{\perp}$ *and by* π *the orthogonal projector onto* $\overline{\mathcal{E}}^{\perp}$*, the normal variable is said to be quasilinear over the fast manifold when any of the following equivalent properties hold*

$$\tilde{\pi}\mathsf{v} = \tilde{\pi}(\partial_{\mathsf{w}}\mathsf{v})\mathsf{w}, \qquad \mathsf{w} \in \mathcal{O}_{\mathsf{w}}, \tag{3.8}$$

$$\pi\mathsf{w} = \pi(\partial_{\mathsf{v}}\mathsf{w})\mathsf{v}, \qquad \mathsf{v} \in \mathcal{O}_{\mathsf{v}}. \tag{3.9}$$

Proof. We have to establish that (3.8) and (3.9) are equivalent. However, $\tilde{\pi}\mathsf{v} = \tilde{\pi}(\partial_{\mathsf{w}}\mathsf{v})\mathsf{w}$ if and only if $\mathsf{v} - (\partial_{\mathsf{w}}\mathsf{v})\mathsf{w} \in \mathcal{E}$ and by definition (3.7) of $\overline{\mathcal{E}}$ this is equivalent to $(\partial_{\mathsf{w}}\mathsf{v})^{-1}\big(\mathsf{v} - (\partial_{\mathsf{w}}\mathsf{v})\mathsf{w}\big) \in \overline{\mathcal{E}}$. Therefore (3.8) is equivalent to $\mathsf{w} - (\partial_{\mathsf{v}}\mathsf{w})\mathsf{v} \in \overline{\mathcal{E}}$ that is to $\pi\mathsf{w} = \pi(\partial_{\mathsf{v}}\mathsf{w})\mathsf{v}$ which is precisely (3.9). $\square$

Proposition 3.9. *Let* $\mathsf{v} \to \mathsf{w}$ *be a normal change of variable, assume that* w *is quasilinear over the fast manifold as in Definition 3.8 and that* $\overline{\mathcal{E}}^{\perp}$ *is a fixed subspace of* $\mathbb{R}^n$*. Then the source term of the normal form* $\overline{\Omega}$ *satisfy properties* $(\overline{\mathsf{S}}_5)$*-*$(\overline{\mathsf{S}}_7)$*, that is properties* (S_5)*-*(S_7) *rewritten in terms of overbar quantities.*

Proof. We first note that $\overline{\mathcal{E}}$ is a fixed subspace by assumption and that $\overline{\Omega} \in \overline{\mathcal{E}}^{\perp}$ since $\overline{\Omega} = (\partial_{\mathsf{w}}\mathsf{v})^t\widetilde{\Omega}$, $\widetilde{\Omega} \in \mathcal{E}^{\perp}$ from (S_5) and $\overline{\mathcal{E}}^{\perp} = (\partial_{\mathsf{w}}\mathsf{v})^t\mathcal{E}^{\perp}$ by definition (3.7) of $\overline{\mathcal{E}}$.

Moreover, $\overline{\Omega}(\mathsf{w}) = 0$ if and only if $\widetilde{\Omega}(\mathsf{v}) = 0$ and if and only if $\mathsf{v} \in \mathcal{E}$ from (S_5). Furthermore $\mathsf{v} \in \mathcal{E}$ if and only if $\tilde{\pi}\mathsf{v} = 0$ and if and only if $(\partial_{\mathsf{w}}\mathsf{v})\mathsf{w} \in \mathcal{E}$ using now relation (3.8). This last relation $(\partial_{\mathsf{w}}\mathsf{v})\mathsf{w} \in \mathcal{E}$ is then equivalent to $\mathsf{w} \in (\partial_{\mathsf{w}}\mathsf{v})^{-1}\mathcal{E} = \overline{\mathcal{E}}$ by definition (3.7). We have thus established that $\overline{\Omega}(\mathsf{w}) = 0$ if and only if $\mathsf{w} \in \overline{\mathcal{E}}$. On the other hand, we note that

$$\big\langle \mathsf{w}, \overline{\Omega}(\mathsf{w})\big\rangle = \big\langle \mathsf{w}, (\partial_{\mathsf{w}}\mathsf{v})^t\widetilde{\Omega}(\mathsf{v})\big\rangle = \big\langle \tilde{\pi}(\partial_{\mathsf{w}}\mathsf{v})\mathsf{w}, \widetilde{\Omega}(\mathsf{v})\big\rangle = \big\langle \tilde{\pi}\mathsf{v}, \widetilde{\Omega}(\mathsf{v})\big\rangle = \big\langle \mathsf{v}, \widetilde{\Omega}(\mathsf{v})\big\rangle,$$

so that $\big\langle \mathsf{w}, \overline{\Omega}(\mathsf{w})\big\rangle = 0$ if and only if $\big\langle \mathsf{v}, \widetilde{\Omega}(\mathsf{v})\big\rangle = 0$ and if and only if $\widetilde{\Omega}(\mathsf{v}) = 0$ from (S_5), and if and only if $\overline{\Omega}(\mathsf{w}) = 0$ by definition of $\overline{\Omega}$ and $(\overline{\mathsf{S}}_5)$ is established.

In order to establish $(\overline{\mathsf{S}}_6)$, we note that if $\mathsf{w} \in \overline{\mathcal{E}}$, then $\overline{\Omega}(\mathsf{w}) = 0$, in such a way that

$$\partial_{\mathsf{w}}\overline{\Omega}(\mathsf{w}) = \big(\partial_{\mathsf{w}}\mathsf{v}(\mathsf{w})\big)^t \partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v})\partial_{\mathsf{w}}\mathsf{v}(\mathsf{w}).$$

This relation now yields that $\partial_{\mathsf{w}}\overline{\Omega}(\mathsf{w})$ is symmetric since $v \in \mathcal{E}$ and $\partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v})$ is symmetric from (S_6), and moreover that $N\big(\partial_{\mathsf{w}}\overline{\Omega}(\mathsf{w})\big) = \big(\partial_{\mathsf{w}}\mathsf{v}(\mathsf{w})\big)^{-1} N\big(\partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v})\big) = \big(\partial_{\mathsf{w}}\mathsf{v}(\mathsf{w})\big)^{-1}\mathcal{E}$ from (S_6) so that finally $N\big(\partial_{\mathsf{w}}\overline{\Omega}(\mathsf{w})\big) = \overline{\mathcal{E}}$ by definition (3.7) of $\overline{\mathcal{E}}$ and $(\overline{\mathsf{S}}_6)$ is established.

Finally, $(\overline{\mathsf{S}}_7)$ is a direct consequence of the identity $\big\langle \mathsf{w}, \overline{\Omega}(\mathsf{w})\big\rangle = \big\langle \mathsf{v}, \widetilde{\Omega}(\mathsf{v})\big\rangle$ and of (S_7) so that the proof is complete. $\square$

All properties of the entropic symmetrized form (S_1)-(S_7) may thus be transferred to the normal form, except for the conservativity of convective fluxes and the—eventual—presence of quadratic residuals $\overline{\mathsf{b}}$.

Corollary 3.10. *Assume that the nullspace invariance property* (N) *holds and that* $\mathsf{v} \to \mathsf{w}$ *is a normal change of variable. Assume that the normal variable* w *is quasilinear over the fast manifold* $\overline{\mathcal{E}}^{\perp}$ *and that* $\overline{\mathcal{E}}^{\perp}$ *is a fixed subspace. Then the following properties* $(\overline{\mathsf{S}}_1)$-$(\overline{\mathsf{S}}_7)$ *hold and the quadratic residual* $\overline{\mathsf{b}}$ *is in the form* (3.6).

$(\overline{\mathsf{S}}_1)$ *The matrix* $\overline{\mathsf{A}}_0(\mathsf{w})$ *has the block structure* (N_1) *and is symmetric positive definite for* $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$.

$(\overline{\mathsf{S}}_2)$ *The matrices* $\overline{\mathsf{A}}_i(\mathsf{w})$, $i \in C$, *are symmetric for* $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$.

$(\overline{\mathsf{S}}_3)$ *The matrices* $\overline{\mathsf{B}}_{ij}(\mathsf{w})$ *have the block structure* (N_1) *and satisfy* $\overline{\mathsf{B}}_{ij}^{t}(\mathsf{w}) = \overline{\mathsf{B}}_{ji}(\mathsf{w})$ *for* $i, j \in C$ *and* $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$.

$(\overline{\mathsf{S}}_4)$ *The matrix* $\overline{\mathsf{B}}^{\mathrm{II},\mathrm{II}}(\mathsf{w}, \boldsymbol{\xi}) = \sum_{i,j \in \mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{II},\mathrm{II}}(\mathsf{w}) \xi_i \xi_j$ *is positive definite for* $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$ *and* $\boldsymbol{\xi} \in \Sigma^{d-1}$.

$(\overline{\mathsf{S}}_5)$ *There exists a fixed vector space* $\overline{\mathcal{E}} \subset \mathbb{R}^n$ *such that* $\overline{\Omega}(\mathsf{w}) \in \overline{\mathcal{E}}^{\perp}$ *for* $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$, *and* $\overline{\Omega}(\mathsf{w}) = 0$ *if and only if* $\mathsf{w} \in \overline{\mathcal{E}}$ *and if and only if* $\langle \mathsf{w}, \overline{\Omega}(\mathsf{w}) \rangle = 0$.

$(\overline{\mathsf{S}}_6)$ *If* $\overline{\Omega}(\mathsf{w}) = 0$, *then* $\partial_{\mathsf{w}} \overline{\Omega}(\mathsf{w}) = \big(\partial_{\mathsf{w}} \overline{\Omega}(\mathsf{w})\big)^t$ *and* $N\big(\partial_{\mathsf{w}} \overline{\Omega}(\mathsf{w})\big) = \overline{\mathcal{E}}$.

$(\overline{\mathsf{S}}_7)$ *We have* $\langle \mathsf{w}, \overline{\Omega}(\mathsf{w}) \rangle \leq 0$ *for* $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$.

We finally investigate the situation where the *quasilinear* structure of the source term $\widetilde{\Omega}$ is transfered to the source term $\overline{\Omega}$ of the normal variable.

Lemma 3.11. *Keep the assumptions of Proposition 3.9 and assume that the symmetrized source term* $\widetilde{\Omega}$ *is in quasilinear form as in Definition 3.4. Then the source term* $\overline{\Omega}$ *is also in quasilinear form*

$$\overline{\Omega}(\mathsf{w}) = -\,\overline{\mathsf{L}}(\mathsf{w})\,\mathsf{w}, \tag{3.10}$$

with

$$\overline{\mathsf{L}} = (\partial_{\mathsf{w}} \mathsf{v})^t\,\widetilde{\mathsf{L}}\,(\partial_{\mathsf{w}} \mathsf{v}), \tag{3.11}$$

and the matrix $\overline{\mathsf{L}}$ *is symmetric positive semi-definite with* $N\big(\overline{\mathsf{L}}(\mathsf{w})\big) = \overline{\mathcal{E}}$ *for* $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$.

Proof. The source term of the normal form $\overline{\Omega}$ is given by $\overline{\Omega} = (\partial_{\mathsf{w}} \mathsf{v})^t \widetilde{\Omega}$ so that we have $\overline{\Omega} = -(\partial_{\mathsf{w}} \mathsf{v})^t \widetilde{\mathsf{L}} \mathsf{v}$. Since $\widetilde{\mathsf{L}} \tilde{\pi} = \widetilde{\mathsf{L}}$ we next deduce using (3.8) that $\overline{\Omega} = -(\partial_{\mathsf{w}} \mathsf{v})^t \widetilde{\mathsf{L}} (\partial_{\mathsf{w}} \mathsf{v}) \mathsf{w} = -\overline{\mathsf{L}}(\mathsf{w}) \mathsf{w}$. It is then easily obtained that $\overline{\mathsf{L}}(\mathsf{w}) = (\partial_{\mathsf{w}} \mathsf{v})^t \widetilde{\mathsf{L}} (\partial_{\mathsf{w}} \mathsf{v})$ is symmetric positive semi-definite with nullspace $\overline{\mathcal{E}}$ and range $\overline{\mathcal{E}}^{\perp}$. □

In the following, we will use a compatibility relation between the mass matrix $\overline{\mathsf{A}}_0$ and the fast manifold $\overline{\mathcal{E}}^{\perp}$ which may be written $\overline{\mathsf{A}}_0 \pi = \pi \overline{\mathsf{A}}_0$ where π is the orthogonal projector onto the fast manifold $\overline{\mathcal{E}}^{\perp}$. This commutation property $\overline{\mathsf{A}}_0 \pi = \pi \overline{\mathsf{A}}_0$ will be especially useful in order to estimate the 'fast component' $\pi \mathsf{w} / \epsilon$ of the normal variable [24]. Equivalent properties are presented in the following lemma which is easily established.

Lemma 3.12. *Let* $\overline{\mathsf{A}}_0 \in \mathbb{R}^{n,n}$ *be symmetric positive definite and* $\overline{\mathcal{E}} \subset \mathbb{R}^n$ *be a linear subspace of* $\mathbb{R}^n$. *Denoting by* π *the orthogonal projector onto* $\overline{\mathcal{E}}^{\perp}$, *the following properties are equivalent.*

(i) $\overline{\mathsf{A}}_0 \pi = \pi \overline{\mathsf{A}}_0$ $\qquad (ii)$ $\overline{\mathsf{A}}_0 \overline{\mathcal{E}}^{\perp} \subset \overline{\mathcal{E}}^{\perp}$ $\qquad (iii)$ $\overline{\mathsf{A}}_0 \overline{\mathcal{E}} \subset \overline{\mathcal{E}}$.

Taking into account that $\overline{\mathsf{A}}_0$ is invertible we also obtain equivalent conditions with equality signs so that (i)–(iii) holds if and only if $\overline{\mathsf{A}}_0 \overline{\mathcal{E}} = \overline{\mathcal{E}}$ for instance.

3.3 Natural entropic symmetrized form

We evaluate in this section the natural entropic symmetrized form for the system of partial differential equations modeling fluids out of thermodynamic equilibrium (2.1)–(2.4). We use the mathematical entropy $\sigma = -\mathcal{S}/r$ where the $1/r$ factor is introduced for convenience. For this particular system of partial differential equations we have $n = d+3$, the velocity components of all quantities in $\mathbb{R}^{d+3}$ are denoted as vectors of $\mathbb{R}^d$ and the corresponding partitioning is also used for matrices.

Theorem 3.13. *Assume that* $(\mathsf{T}_1)(\mathsf{T}_2)$ *and* $(\mathsf{Tr}_1)(\mathsf{Tr}_2)$ *hold. Then the function* $\sigma = -\mathcal{S}/r$ *is a mathematical entropy for the system* (2.1)–(2.4) *and the corresponding entropic variable is*

$$\mathsf{v} = (\partial_{\mathsf{u}}\sigma)^t = \frac{1}{r}\Big(\frac{g_{\mathrm{tr}}}{T_{\mathrm{tr}}} + \frac{g_{\mathrm{in}}}{T_{\mathrm{in}}} - \frac{\frac{1}{2}|\boldsymbol{v}|^2}{T_{\mathrm{tr}}},\ \frac{\boldsymbol{v}}{T_{\mathrm{tr}}},\ \frac{1}{T_{\mathrm{tr}}} - \frac{1}{T_{\mathrm{in}}},\ -\frac{1}{T_{\mathrm{tr}}}\Big)^t. \tag{3.12}$$

The map $\mathsf{u} \to \mathsf{v}$ *is a* $C^{\varkappa-1}$ *diffeomorphism from* $\mathcal{O}_{\mathsf{u}}$ *given in* (2.28) *onto the open set* $\mathcal{O}_{\mathsf{v}}$ *given by*

$$\mathcal{O}_{\mathsf{v}} = \{\mathsf{v} \in \mathbb{R}^n;\ \mathsf{v}_{d+2} + \mathsf{v}_{d+3} < 0,\ \mathsf{v}_{d+3} < 0\}. \tag{3.13}$$

The system written in terms of the entropic variable v *is of the symmetric form with a source term in quasilinear form*

$$\widetilde{\mathsf{A}}_0(\mathsf{v})\partial_t \mathsf{v} + \sum_{i\in\mathcal{D}} \widetilde{\mathsf{A}}_i(\mathsf{v})\partial_i \mathsf{v} - \epsilon_d \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}_{ij}(\mathsf{v})\partial_j \mathsf{v}\big) + \frac{1}{\epsilon}\widetilde{\mathsf{L}}(\mathsf{v})\mathsf{v} = 0, \tag{3.14}$$

where $\widetilde{\mathsf{A}}_0 = \partial_{\mathsf{v}}\mathsf{u}$, $\widetilde{\mathsf{A}}_i = \mathsf{A}_i\partial_{\mathsf{v}}\mathsf{u}$, $\widetilde{\mathsf{B}}_{ij} = \mathsf{B}_{ij}\partial_{\mathsf{v}}\mathsf{u}$, $\widetilde{\Omega} = \Omega = -\widetilde{\mathsf{L}}(\mathsf{v})\mathsf{v}$ *and the coefficients have regularity* $C^{\varkappa-1}$. *The matrix* $\widetilde{\mathsf{A}}_0$ *is given by*

$$\widetilde{\mathsf{A}}_0 = \begin{bmatrix} \rho & & & Sym \\ \rho\boldsymbol{v} & \rho\boldsymbol{v}\otimes\boldsymbol{v} + \rho r T_{\mathrm{tr}}\boldsymbol{I} & & \\ \rho e_{\mathrm{in}} & \rho e_{\mathrm{in}}\boldsymbol{v}^t & \rho(e_{\mathrm{in}})^2 + \rho r c_{\mathrm{in}} T_{\mathrm{in}}^2 & \\ \rho e_{\mathrm{tl}} & \rho h_{\mathrm{tl}}\boldsymbol{v}^t & \rho e_{\mathrm{in}} e_{\mathrm{tl}} + \rho r c_{\mathrm{in}} T_{\mathrm{in}}^2 & \widetilde{\Upsilon}_{\mathrm{tl}} \end{bmatrix}, \tag{3.15}$$

where $\widetilde{\Upsilon}_{\mathrm{tl}} = \rho(e_{\mathrm{tl}})^2 + \rho r T_{\mathrm{tr}}|\boldsymbol{v}|^2 + \rho r c_{\mathrm{in}} T_{\mathrm{in}}^2 + \rho r c_{\mathrm{v,tr}} T_{\mathrm{tr}}^2$ *and* $e_{\mathrm{tl}} = e + \frac{1}{2}|\boldsymbol{v}|^2 = e_{\mathrm{tr}} + e_{\mathrm{in}} + \frac{1}{2}|\boldsymbol{v}|^2$. *Since* $\widetilde{\mathsf{A}}_0$ *is symmetric, we only give its left lower triangular part and write "Sym" in the upper triangular part. Denoting by* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)^t$ *an arbitrary vector of* $\mathbb{R}^d$ *and letting* $\widetilde{\mathsf{A}} = \sum_{i\in\mathcal{D}} \xi_i \widetilde{\mathsf{A}}_i$, *we have*

$$\widetilde{\mathsf{A}} = \boldsymbol{v}\cdot\boldsymbol{\xi}\,\widetilde{\mathsf{A}}_0 + \rho r T_{\mathrm{tr}} \begin{bmatrix} 0 & & & Sym \\ \boldsymbol{\xi} & \boldsymbol{\xi}\otimes\boldsymbol{v} + \boldsymbol{v}\otimes\boldsymbol{\xi} & & \\ 0 & e_{\mathrm{in}}\boldsymbol{\xi}^t & 0 & \\ \boldsymbol{v}\cdot\boldsymbol{\xi} & \boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{v}^t + h_{\mathrm{tl}}\boldsymbol{\xi}^t & \boldsymbol{v}\cdot\boldsymbol{\xi} e_{\mathrm{in}} & 2\boldsymbol{v}\cdot\boldsymbol{\xi} h_{\mathrm{tl}} \end{bmatrix}, \tag{3.16}$$

where $h_{\mathrm{tl}} = e_{\mathrm{tl}} + r T_{\mathrm{tr}} = h + \frac{1}{2}|\boldsymbol{v}|^2$ *denotes the total enthalpy per unit mass. Moreover, we have the decomposition*

$$\widetilde{\mathsf{B}}_{ij} = r\widetilde{\mathsf{B}}^{\lambda}\delta_{ij} + \bar{\eta} r T_{\mathrm{tr}}\,\widetilde{\mathsf{B}}^{\eta}_{ij}, \tag{3.17}$$

where

$$\widetilde{\mathsf{B}}^{\lambda} = \begin{bmatrix} 0 & & & Sym \\ 0_{d,1} & 0_{d,d} & & \\ 0 & 0_{1,d} & T_{\mathrm{in}}^2\bar{\lambda}_{\mathrm{in,in}} & T_{\mathrm{tr}}^2\bar{\lambda}_{\mathrm{in,tr}} + T_{\mathrm{in}}^2\bar{\lambda}_{\mathrm{in,in}} \\ 0 & 0_{1,d} & T_{\mathrm{in}}^2(\bar{\lambda}_{\mathrm{tr,in}} + \bar{\lambda}_{\mathrm{in,in}}) & T_{\mathrm{tr}}^2(\bar{\lambda}_{\mathrm{tr,tr}} + \bar{\lambda}_{\mathrm{in,tr}}) + T_{\mathrm{in}}^2(\bar{\lambda}_{\mathrm{tr,in}} + \bar{\lambda}_{\mathrm{in,in}}) \end{bmatrix}, \tag{3.18}$$

and denoting by $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)^t$ *and* $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_d)^t$ *arbitrary vectors of* $\mathbb{R}^d$*, the matrices* $\widetilde{\mathsf{B}}_{ij}^{\eta}$*,* $i,j \in \mathcal{D}$*, are given by*

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \widetilde{\mathsf{B}}_{ij}^{\eta} = \begin{bmatrix} 0 & 0_{1,d} & 0 & 0 \\ 0_{d,1} & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\mathbb{I}_d + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} - \frac{2}{d'}\boldsymbol{\xi}\otimes\boldsymbol{\zeta} & 0_{d,1} & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\xi}\,\boldsymbol{\zeta} - \frac{2}{d'}\boldsymbol{v}\cdot\boldsymbol{\zeta}\,\boldsymbol{\xi} \\ 0 & 0_{1,d} & 0 & 0 \\ 0 & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\boldsymbol{v}^t + \boldsymbol{v}\cdot\boldsymbol{\zeta}\,\boldsymbol{\xi}^t - \frac{2}{d'}\boldsymbol{v}\cdot\boldsymbol{\xi}\,\boldsymbol{\zeta}^t & 0 & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\boldsymbol{v}\cdot\boldsymbol{v} + (1-\frac{2}{d'})\boldsymbol{v}\cdot\boldsymbol{\xi}\,\boldsymbol{v}\cdot\boldsymbol{\zeta} \end{bmatrix}. \tag{3.19}$$

Finally, the equilibrium manifold is given by

$$\mathcal{E} = \mathbb{R}\times\mathbb{R}^d\times\{0\}\times\mathbb{R}, \tag{3.20}$$

the fast manifold by $\mathcal{E}^{\perp} = \mathbb{R}\mathsf{e}_{d+2}$*, the projector by* $\tilde{\pi} = \mathsf{e}_{d+2}\otimes\mathsf{e}_{d+2}$ *where* $\mathsf{e}_1, \ldots, \mathsf{e}_n$ *denote the basis vectors of* $\mathbb{R}^n$*, and the matrix* $\widetilde{\mathsf{L}}$ *associated with the source term* $\widetilde{\Omega}$ *by*

$$\widetilde{\mathsf{L}}(\mathsf{v}) = r\frac{\rho c_{\mathrm{in}}}{\overline{\tau}_{\mathrm{in}}} T_{\mathrm{in}} T_{\mathrm{tr}}\, \mathsf{e}_{d+2}\otimes\mathsf{e}_{d+2}. \tag{3.21}$$

Proof. The explicit calculation of the natural symmetrized system is long but presents no difficulty. This establishes in particular all symmetry properties for $\widetilde{\mathsf{A}}_0$, $\widetilde{\mathsf{A}}_i$, $i \in \mathcal{D}$, as well as the reciprocity relations for $\widetilde{\mathsf{B}}_{ij}$, $i,j \in \mathcal{D}$, and the integer regularity class $\varkappa - 1$ of the matrix coefficients.

Letting then $\mathsf{x} = (\mathsf{x}_\rho, \mathbf{x}_{\boldsymbol{v}}, \mathsf{x}_{\mathrm{in}}, \mathsf{x}_{\mathrm{tl}})^t$, with $\mathbf{x}_{\boldsymbol{v}} = (\mathsf{x}_{1+1}, \ldots, \mathsf{x}_{1+d})^t$, the quadratic form associated with $\widetilde{\mathsf{A}}_0$ is evaluated to be

$$\langle \widetilde{\mathsf{A}}_0\mathsf{x}, \mathsf{x}\rangle = \rho(\mathsf{x}_\rho + \boldsymbol{v}\cdot\mathbf{x}_{\boldsymbol{v}} + e_{\mathrm{in}}\mathsf{x}_{\mathrm{in}} + e_{\mathrm{tl}}\mathsf{x}_{\mathrm{tl}})^2 + \rho r T_{\mathrm{tr}}|\mathbf{x}_{\boldsymbol{v}} + \boldsymbol{v}\mathsf{x}_{\mathrm{tl}}|^2 + \rho r c_{\mathrm{in}} T_{\mathrm{in}}^2(\mathsf{x}_{\mathrm{in}} + \mathsf{x}_{\mathrm{tl}})^2 + \rho r c_{\mathrm{v,tr}} T_{\mathrm{tr}}^2 \mathsf{x}_{\mathrm{tl}}^2,$$

so that $\widetilde{\mathsf{A}}_0$ is positive definite.

The matrix $\widetilde{\mathsf{B}}^{\lambda}$ is positive semi-definite from Lemma 2.1 so that for any $\boldsymbol{\xi} \in \mathbb{R}^d$, the matrix $\sum_{i,j\in\mathcal{D}} \xi_i\xi_j\delta_{ij}\widetilde{\mathsf{B}}^{\lambda} = (\sum_{i\in\mathcal{D}}\xi_i^2)\widetilde{\mathsf{B}}^{\lambda}$ is also positive semi-definite. Letting next $\widetilde{\mathsf{B}}^{\eta} = \sum_{i,j\in\mathcal{D}}\xi_i\xi_j\widetilde{\mathsf{B}}_{ij}^{\eta}$ for $\boldsymbol{\xi} \in \Sigma^{d-1}$ and writting $\mathsf{x} = (\mathsf{x}_1, \mathbf{x}_{\boldsymbol{v}}, \mathsf{x}_{\mathrm{in}}, \mathsf{x}_{\mathrm{tl}})^t$ for $\mathsf{x} \in \mathbb{R}^n$, we have

$$\big\langle \widetilde{\mathsf{B}}^{\eta}\mathsf{x}, \mathsf{x}\big\rangle = \big(1 - \tfrac{2}{d'}\big)\big(\boldsymbol{\xi}\cdot(\mathbf{x}_{\boldsymbol{v}} + \boldsymbol{v}\mathsf{x}_{\mathrm{tl}})\big)^2 + |\boldsymbol{\xi}|^2\big|\mathbf{x}_{\boldsymbol{v}} + \boldsymbol{v}\mathsf{x}_{\mathrm{tl}}\big|^2,$$

so that $\widetilde{\mathsf{B}}^{\eta}$ is positive semi-definite with nullspace $N(\widetilde{\mathsf{B}}^{\eta}) = \{\mathsf{x}\in\mathbb{R}^n;\ \mathbf{x}_{\boldsymbol{v}} + \boldsymbol{v}\mathsf{x}_{\mathrm{tl}} = 0\}$ since $d' \geq 2$.

The source term reads $\widetilde{\Omega} = \big(0, \mathbf{0}, \overline{\omega}_{\mathrm{in}}, 0\big)^t$ where $\overline{\omega}_{\mathrm{in}} = \rho c_{\mathrm{in}}(T_{\mathrm{tr}} - T_{\mathrm{in}})/\overline{\tau}_{\mathrm{in}}$ and the equilibrium manifold has been defined from (3.20) so that $\mathcal{E}^{\perp} = \{0\}\times\{0\}^d\times\mathbb{R}\times\{0\} = \mathbb{R}\mathsf{e}_{d+2}$ and $\widetilde{\Omega} \in \mathcal{E}^{\perp}$ by construction. It is then easily checked that $\widetilde{\Omega}(\mathsf{v}) = -\widetilde{L}(\mathsf{v})\mathsf{v}$, where $\widetilde{\mathsf{L}}(\mathsf{v}) = r\frac{\rho c_{\mathrm{in}}}{\overline{\tau}_{\mathrm{in}}}T_{\mathrm{in}}T_{\mathrm{tr}}\mathsf{e}_{d+2}\otimes\mathsf{e}_{d+2}$ and $\mathsf{e}_1, \ldots, \mathsf{e}_n$ denote the basis vectors of $\mathbb{R}^n$, is positive semi-definite with nullspace $N\big(\widetilde{L}(\mathsf{v})\big) = \mathcal{E}$ and the proof is complete. □

3.4 Normal form

We first investigate the nullspace invariance property for the symmetrized system (3.14) modeling fluids out of thermodynamic equilibrium and then evaluate a convenient normal form.

Lemma 3.14. *For any* $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$ *and* $\boldsymbol{\xi} \in \Sigma^{d-1}$ *the nullspace of the matrix* $\widetilde{\mathsf{B}}(\mathsf{v}, \boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}}\widetilde{\mathsf{B}}_{ij}(\mathsf{v})\xi_i\xi_j$ *is given by*

$$N(\widetilde{\mathsf{B}}) = \mathbb{R}(1, \mathbf{0}, 0, 0)^t,$$

and we have $\widetilde{\mathsf{B}}_{ij}(\mathsf{v})N(\widetilde{\mathsf{B}}) = 0$ *for* $i,j \in \mathcal{D}$ *and* $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$*.*

Proof. From the proof of Theorem 3.13, letting $\mathsf{x} = \big(\mathsf{x}_\rho, \mathbf{x}_{\boldsymbol{v}}, \mathsf{x}_{\mathrm{in}}, \mathsf{x}_{\mathrm{tl}}\big)^t$, with $\mathbf{x}_{\boldsymbol{v}} = \big(\mathsf{x}_{1+1}, \ldots, \mathsf{x}_{1+d}\big)^t$, we obtain

$$\big\langle \widetilde{\mathsf{B}}\mathsf{x}, \mathsf{x}\big\rangle = \bar{\eta} r T_{\mathrm{tr}} \tfrac{d'-2}{d}\big(\boldsymbol{\xi}\cdot(\mathbf{x}_{\boldsymbol{v}} + \boldsymbol{v}\mathsf{x}_{\mathrm{tl}})\big)^2 + \bar{\eta} r T_{\mathrm{tr}}|\boldsymbol{\xi}|^2\big|\mathbf{x}_{\boldsymbol{v}} + \boldsymbol{v}\mathsf{x}_{\mathrm{tl}}\big|^2 + |\boldsymbol{\xi}|^2\big\langle \widetilde{\mathsf{B}}^{\lambda}\mathsf{x}, \mathsf{x}\big\rangle.$$

Assuming that $\big\langle \widetilde{\mathsf{B}}\mathsf{x}, \mathsf{x}\big\rangle = 0$ and $|\boldsymbol{\xi}| = 1$, we obtain that $\big\langle \widetilde{\mathsf{B}}^{\lambda}\mathsf{x}, \mathsf{x}\big\rangle = 0$ and $\big|\mathbf{x}_{\boldsymbol{v}} + \boldsymbol{v}\mathsf{x}_{\mathrm{tl}}\big| = 0$ since $d' \geq 2$. From the structure of $\widetilde{\mathsf{B}}^{\lambda}$ and Lemma 2.1 we deduce that $\mathsf{x}_{\mathrm{in}} = 0$ and $\mathsf{x}_{\mathrm{tl}} = 0$ and from $\big|\mathbf{x}_{\boldsymbol{v}} + \boldsymbol{v}\mathsf{x}_{\mathrm{tl}}\big| = 0$ we next obtain that $\mathbf{x}_{\boldsymbol{v}} = 0$ so that $\mathsf{x}_{1+i} = 0$ for $1 \leq i \leq d$. We have thus established that $N(\widetilde{\mathsf{B}})$ is spanned by $(1, \mathbf{0}, 0, 0)^t$ and it is easily checked that $\widetilde{\mathsf{B}}_{ij}(\mathsf{v})N(\widetilde{\mathsf{B}}) = 0$, for $i,j \in \mathcal{D}$, $\mathsf{v} \in \mathcal{O}_{\mathsf{v}}$. □

Since $N(\widetilde{\mathsf{B}})$ is spanned by $(1,\mathbf{0},0,0)^t$, we may use the identity matrix $\mathbb{I}_n$ as the auxiliary matrix P in Theorem 3.7 and the corresponding auxiliary variables u' and v' then coincide with u and v, respectively. From Theorem 3.7 and the relations (2.26) and (3.12) all normal variables are thus in the form $\big(\mathcal{F}_{\rm I}(\rho),\mathcal{F}_{\rm II}(\boldsymbol{v},T_{\rm in},T_{\rm tr})\big)^t$ where $\mathcal{F}_{\rm I}$ and $\mathcal{F}_{\rm II}$ are diffeomorphisms in $\mathbb{R}$ and $\mathbb{R}^{d+2}$, respectively. The natural variable z is in particular a normal variable but for convenience we select the normal variable

$$\mathsf{w}=\Big(\rho,\boldsymbol{v},\frac{1}{T_{\rm tr}}-\frac{1}{T_{\rm in}},-\frac{1}{T}\Big)^t. \tag{3.22}$$

The third component of w indeed goes to zero with the relaxation time and the other components $\big(\rho,\boldsymbol{v},-\frac{1}{T}\big)^t$ will converge towards the corresponding normal variable at thermodynamic equilibrium $\mathsf{w}_{\rm e}=\big(\rho_{\rm e},\boldsymbol{v}_{\rm e},-\frac{1}{T_{\rm e}}\big)^t$. Moreover the corresponding normal form guarantees the important commutation relation between the "mass" matrix $\overline{\mathsf{A}}_0$ and the orthogonal projector π onto $\overline{\mathcal{E}}^{\perp}$. We evaluate explicitly in the following theorem the corresponding equations in normal form.

Theorem 3.15. *Assume that* $(\mathsf{T}_1)(\mathsf{T}_2)$ *and* $(\mathsf{Tr}_1)(\mathsf{Tr}_2)$ *hold. Then the map* $\mathsf{v}\to\mathsf{w}$ *is a* $C^{\varkappa-1}$ *diffeomorphism from* $\mathcal{O}_{\mathsf{v}}$ *onto the open set* $\mathcal{O}_{\mathsf{w}}$ *given by*

$$\mathcal{O}_{\mathsf{w}}=(0,\infty)\times\mathbb{R}^d\times\mathbb{R}\times(-\infty,0). \tag{3.23}$$

The system written in the w *variable is in the normal form with a source term in quasilinear form*

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\mathsf{w}+\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\mathsf{w}-\epsilon_d\sum_{i,j\in\mathcal{D}}\partial_i\big(\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_j\mathsf{w}\big)+\frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\mathsf{w}=\epsilon_d\overline{\mathsf{b}}(\mathsf{w},\partial_{\mathsf{x}}\mathsf{w}_{\rm II}), \tag{3.24}$$

where $\overline{\mathsf{A}}_0=(\partial_{\mathsf{w}}\mathsf{v})^t\,\widetilde{\mathsf{A}}_0\,\partial_{\mathsf{w}}\mathsf{v}$, $\overline{\mathsf{A}}_i=(\partial_{\mathsf{w}}\mathsf{v})^t\,\widetilde{\mathsf{A}}_i\,\partial_{\mathsf{w}}\mathsf{v}$, $\overline{\mathsf{B}}_{ij}=(\partial_{\mathsf{w}}\mathsf{v})^t\,\widetilde{\mathsf{B}}_{ij}\,\partial_{\mathsf{w}}\mathsf{v}$, $\overline{\Omega}=(\partial_{\mathsf{w}}\mathsf{v})^t\,\Omega=-\overline{\mathsf{L}}(\mathsf{w})\mathsf{w}$, *have regularity* $\varkappa-1$ *and* $\overline{\mathsf{b}}=-\sum_{i,j\in\mathcal{D}}\partial_i(\partial_{\mathsf{w}}\mathsf{v})^t\,\widetilde{\mathsf{B}}_{ij}\;\partial_{\mathsf{w}}\mathsf{v}\,\partial_j\mathsf{w}$. *The matrix* $\overline{\mathsf{A}}_0$ *is given by*

$$\overline{\mathsf{A}}_0=\begin{bmatrix}\frac{1}{\rho} & & & Sym\\ 0_{d,1} & \frac{\rho\boldsymbol{I}}{rT_{\rm tr}} & & \\ 0 & 0_{1,d} & \overline{\Upsilon}_{\rm in} & \\ 0 & 0_{1,d} & 0 & \overline{\Upsilon}_{\rm tl}\end{bmatrix},$$

where

$$\overline{\Upsilon}_{\rm in}=\rho c_{\rm tr}c_{\rm in}T_{\rm tr}^2T_{\rm in}^2\,\frac{c_{\rm tr}T_{\rm in}^2+c_{\rm in}T_{\rm tr}^2}{r(c_{\rm tr}T_{\rm tr}^2+c_{\rm in}T_{\rm in}^2)^2},\qquad \overline{\Upsilon}_{\rm tl}=\rho T^4\frac{c_{\rm v}^2}{r(c_{\rm tr}T_{\rm tr}^2+c_{\rm in}T_{\rm in}^2)},$$

with $c_{\rm in}$ *evaluated at* $T_{\rm in}$ *and* $c_{\rm v}$ *evaluated at* T. *Denoting by* $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_d)^t$ *an arbitrary vector of* $\mathbb{R}^d$, *the matrices* $\overline{\mathsf{A}}_i$, $i\in\mathcal{D}$, *are given by*

$$\sum_{i\in\mathcal{D}}\xi_i\overline{\mathsf{A}}_i=\overline{\mathsf{A}}_0\boldsymbol{v}\cdot\boldsymbol{\xi}+\begin{bmatrix}0 & & & Sym\\ \boldsymbol{\xi} & 0_{d,d} & & \\ 0 & -\theta_{\rm in}\boldsymbol{\xi}^t & 0 & \\ 0 & \theta_{\rm tl}\boldsymbol{\xi}^t & 0 & 0\end{bmatrix},$$

with

$$\theta_{\rm in}=\frac{\rho T_{\rm tr}T_{\rm in}^2Tc_{\rm in}}{c_{\rm tr}T_{\rm tr}^2+c_{\rm in}T_{\rm in}^2},\qquad \theta_{\rm tl}=\frac{\rho T_{\rm tr}T^2c_{\rm v}}{c_{\rm tr}T_{\rm tr}^2+c_{\rm in}T_{\rm in}^2}.$$

The matrices $\overline{\mathsf{B}}_{ij}$ *have the structure*

$$\overline{\mathsf{B}}_{ij}=\frac{1}{r}\overline{\mathsf{B}}^{\lambda}\delta_{ij}+\frac{\overline{\eta}}{rT_{\rm tr}}\overline{\mathsf{B}}_{ij}^{\eta},$$

where the matrix $\overline{\mathsf{B}}^{\lambda}$ *is given by*

$$\overline{\mathsf{B}}^{\lambda}=\begin{bmatrix}0 & & & Sym\\ 0_{d,1} & 0_{d,d} & & \\ 0 & 0 & \widehat{\lambda}_{\rm in,in} & \\ 0 & 0 & \widehat{\lambda}_{\rm tl,in} & \widehat{\lambda}_{\rm tl,tl}\end{bmatrix},$$

with

$$\widehat{\lambda}_{\mathrm{in,in}} = \frac{T_{\mathrm{tr}}^4 T_{\mathrm{in}}^4}{(c_{\mathrm{tr}}T_{\mathrm{tr}}^2 + c_{\mathrm{in}}T_{\mathrm{in}}^2)^2}\Big(\frac{c_{\mathrm{tr}}^2\overline{\lambda}_{\mathrm{in,in}}}{T_{\mathrm{in}}^2} - \frac{c_{\mathrm{tr}}c_{\mathrm{in}}\overline{\lambda}_{\mathrm{in,tr}}}{T_{\mathrm{in}}^2} - \frac{c_{\mathrm{tr}}c_{\mathrm{in}}\overline{\lambda}_{\mathrm{tr,in}}}{T_{\mathrm{tr}}^2} + \frac{c_{\mathrm{in}}^2\overline{\lambda}_{\mathrm{tr,tr}}}{T_{\mathrm{tr}}^2}\Big),$$

$$\widehat{\lambda}_{\mathrm{in,tl}} = \widehat{\lambda}_{\mathrm{tl,in}} = \frac{c_{\mathrm{v}}T_{\mathrm{tr}}^4 T_{\mathrm{in}}^4 T^2}{(c_{\mathrm{tr}}T_{\mathrm{tr}}^2 + c_{\mathrm{in}}T_{\mathrm{in}}^2)^2}\Big(\frac{c_{\mathrm{tr}}\overline{\lambda}_{\mathrm{in,in}}}{T_{\mathrm{tr}}^2 T_{\mathrm{in}}^2} + \frac{c_{\mathrm{tr}}\overline{\lambda}_{\mathrm{in,tr}}}{T_{\mathrm{in}}^4} - \frac{c_{\mathrm{in}}\overline{\lambda}_{\mathrm{tr,in}}}{T_{\mathrm{tr}}^4} - \frac{c_{\mathrm{in}}\overline{\lambda}_{\mathrm{tr,tr}}}{T_{\mathrm{tr}}^2 T_{\mathrm{in}}^2}\Big),$$

$$\widehat{\lambda}_{\mathrm{tl,tl}} = \frac{c_{\mathrm{v}}^2 T_{\mathrm{tr}}^2 T_{\mathrm{in}}^4 T^4}{(c_{\mathrm{tr}}T_{\mathrm{tr}}^2 + c_{\mathrm{in}}T_{\mathrm{in}}^2)^2}\Big(\frac{\overline{\lambda}_{\mathrm{in,in}}}{T_{\mathrm{tr}}^2 T_{\mathrm{in}}^2} + \frac{\overline{\lambda}_{\mathrm{in,tr}}}{T_{\mathrm{in}}^4} + \frac{\overline{\lambda}_{\mathrm{tr,in}}}{T_{\mathrm{tr}}^2 T_{\mathrm{in}}^2} + \frac{\overline{\lambda}_{\mathrm{tr,tr}}}{T_{\mathrm{in}}^4}\Big).$$

Denoting by $\boldsymbol{\xi} = (\xi_1,\dots,\xi_d)^t$ *and* $\boldsymbol{\zeta} = (\zeta_1,\dots,\zeta_d)^t$ *arbitrary vectors of* $\mathbb{R}^d$*, the matrices* $\overline{\mathsf{B}}_{ij}^{\eta}$*,* $i,j\in\mathcal{D}$*, are given by*

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \overline{\mathsf{B}}_{ij}^{\eta} = \begin{bmatrix} 0 & 0_{1,d} & 0 & 0 \\ 0_d & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\boldsymbol{I} + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} - \frac{2}{d'}\boldsymbol{\xi}\otimes\boldsymbol{\zeta} & 0_{d,1} & 0_{d,1} \\ 0 & 0_{1,d} & 0 & 0 \\ 0 & 0_{1,d} & 0 & 0 \end{bmatrix}.$$

The quadratic residual is given by

$$\overline{\mathsf{b}} = \Big(0, \overline{\mathsf{b}}_{\boldsymbol{v}}, \overline{\mathsf{b}}_{\mathrm{in}}, \overline{\mathsf{b}}_{\mathrm{tl}}\Big)^t$$

where

$$r\overline{\mathsf{b}}_{\boldsymbol{v}} = \sum_{i\in\mathcal{D}} \boldsymbol{\Pi}_i \partial_i\Big(\frac{1}{T_{\mathrm{tr}}}\Big),$$

$$r\overline{\mathsf{b}}_{\mathrm{in}} = \frac{c_{\mathrm{in}}T_{\mathrm{in}}^2}{c_{\mathrm{tr}}T_{\mathrm{tr}}^2 + c_{\mathrm{in}}T_{\mathrm{in}}^2}\sum_{i,j\in\mathcal{D}} \Pi_{ij}\partial_i v_j - \partial_i\Big(\frac{c_{\mathrm{in}}T_{\mathrm{in}}^2}{c_{\mathrm{tr}}T_{\mathrm{tr}}^2 + c_{\mathrm{in}}T_{\mathrm{in}}^2}\Big)(Q_{\mathrm{in},i} + Q_{\mathrm{tr},i}),$$

$$r\overline{\mathsf{b}}_{\mathrm{tl}} = -\frac{c_{\mathrm{v}}T^2}{c_{\mathrm{tr}}T_{\mathrm{tr}}^2 + c_{\mathrm{in}}T_{\mathrm{in}}^2}\sum_{i,j\in\mathcal{D}} \Pi_{ij}\partial_i v_j - \partial_i\Big(\frac{c_{\mathrm{v}}T^2}{c_{\mathrm{tr}}T_{\mathrm{tr}}^2 + c_{\mathrm{in}}T_{\mathrm{in}}^2}\Big)(Q_{\mathrm{in},i} + Q_{\mathrm{tr},i}).$$

The quadratic residual may be written in the form

$$\overline{\mathsf{b}} = \Big(0, \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}^{\mathrm{II,II,II}}(\mathsf{w})\,\partial_i\mathsf{w}_{\mathrm{II}}\,\partial_j\mathsf{w}_{\mathrm{II}}\Big)^t,$$

where $\overline{\mathrm{M}}^{\mathrm{II,II,II}}$ *have regularity* $\varkappa - 2$ *since* c_{in} *is of class* $\varkappa - 1$*. The source term is given by*

$$\overline{\Omega} = \frac{1}{r}\Big(0, \mathbf{0}, \overline{\omega}_{\mathrm{in}}, 0\Big)^t,$$

where $\overline{\omega}_{\mathrm{in}} = \rho c_{\mathrm{in}}(T_{\mathrm{tr}} - T_{\mathrm{in}})/\overline{\tau}_{\mathrm{in}}$*. The equilibrium linear manifold with respect to the normal variable is a fixed subspace given by*

$$\overline{\mathcal{E}} = \mathcal{E} = \mathbb{R}\times\mathbb{R}^d\times\{0\}\times\mathbb{R}, \tag{3.25}$$

and the normal variable w *is quasilinear on the fast manifold* $\overline{\mathcal{E}}^{\perp} = \mathbb{R}\mathsf{e}_{d+2}$ *so that* (3.8) *holds. The source term* $\overline{\Omega}$ *may thus be written in quasilinear form* $\overline{\Omega}(\mathsf{w}) = -\overline{\mathsf{L}}(\mathsf{w})\,\mathsf{w}$ *with*

$$\overline{\mathsf{L}}(\mathsf{w}) = \frac{\rho c_{\mathrm{in}}}{r\overline{\tau}_{\mathrm{in}}} T_{\mathrm{in}}T_{\mathrm{tr}}\,\mathsf{e}_{d+2}\otimes\mathsf{e}_{d+2}, \tag{3.26}$$

where $\mathsf{e}_1,\dots,\mathsf{e}_n$ *denote the basis vectors of* $\mathbb{R}^n$*. Finally, the normal variable is compatible with the fast manifold and*

$$\pi\overline{\mathsf{A}}_0 = \overline{\mathsf{A}}_0\pi. \tag{3.27}$$

Proof. The proof is lengthy and tedious but presents no serious difficulties. □

4 Equations at equilibrium

The equation governing one-temperature fluids are presented in this section. Symmetrized form of the corresponding system of partial differential equations are obtained as well as their links with that of the system out of equilibrium.

4.1 Conservation equations

In order to investigate the fast relaxation limit $\epsilon \to 0$ we will need to establish a stability theorem for the equations governing fluids at thermodynamic equilibrium that are summarized in this section. The equations modeling one-temperature fluids are in the form [18]

$$\partial_t \rho_{\rm e} + \boldsymbol{\nabla}\cdot(\rho_{\rm e}\boldsymbol{v}_{\rm e}) = 0, \tag{4.1}$$

$$\partial_t(\rho_{\rm e}\boldsymbol{v}_{\rm e}) + \boldsymbol{\nabla}\cdot(\rho_{\rm e}\boldsymbol{v}_{\rm e}\otimes\boldsymbol{v}_{\rm e} + p_{\rm e}\boldsymbol{I}) + \boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_{\rm e} = 0, \tag{4.2}$$

$$\partial_t(\mathcal{E}_{\rm e} + \tfrac{1}{2}\rho|\boldsymbol{v}_{\rm e}|^2) + \boldsymbol{\nabla}\cdot\big(\boldsymbol{v}_{\rm e}(\mathcal{E}_{\rm e} + \tfrac{1}{2}\rho|\boldsymbol{v}_{\rm e}|^2 + p_{\rm e})\big) + \boldsymbol{\nabla}\cdot(\boldsymbol{Q}_{\rm e} + \boldsymbol{\Pi}_{\rm e}\cdot\boldsymbol{v}_{\rm e}) = 0, \tag{4.3}$$

where the subscript e denotes thermodynamic equilibrium, $\rho_{\rm e}$ the mass density, $\boldsymbol{v}_{\rm e}$ the fluid velocity, $p_{\rm e}$ the pressure, $\boldsymbol{\Pi}_{\rm e}$ the viscous tensor involving the volume viscosity, $\mathcal{E}_{\rm e}$ the internal energy per unit volume, and $\boldsymbol{Q}_{\rm e}$ the heat flux.

The pressure $p_{\rm e}$ and the internal energy per unit volume $\mathcal{E}_{\rm e}$ are in the form $p_{\rm e} = \rho_{\rm e} r T_{\rm e}$ and $\mathcal{E}_{\rm e} = \rho_{\rm e} e_{\rm e}$ where $e_{\rm e}$ denotes the total internal energy per unit mass. This energy per unit mass $e_{\rm e}(T_{\rm e})$ is given by

$$e_{\rm e} = e_{\rm e,st} + \int_{T_{\rm st}}^{T_{\rm e}} c_{\rm ve}(\theta)\,{\rm d}\theta, \tag{4.4}$$

where $c_{\rm ve}(T_{\rm e}) = c_{\rm v,tr} + c_{\rm in}(T_{\rm e})$ denotes the heat at constant volume per unit mass, $T_{\rm e}$ the equilibrium temperature, $e_{\rm e,st}$ formation energy at the standard temperature and we have $e_{\rm e}(T_{\rm e}) = e_{\rm tr}(T_{\rm e}) + e_{\rm in}(T_{\rm e})$. We will also use in the following the heat at constant pressure per unit mass $c_{p{\rm e}} = c_{\rm ve} + r$ and the formation energy at zero temperature $e_{\rm e}^0 = e_{\rm e}(0)$.

The entropy per unit volume $\mathcal{S}_{\rm e}$ is given by $\mathcal{S}_{\rm e} = \rho_{\rm e} s_{\rm e}$ where $s_{\rm e}$ denotes the entropy per unit mass. This entropy per unit mass $s_{\rm e}(\rho_{\rm e}, T_{\rm e})$ is in the form

$$s_{\rm e} = s_{\rm e,st} + \int_{T_{\rm st}}^{T_{\rm e}} \frac{c_{\rm ve}(\theta)}{\theta}\,{\rm d}\theta - r\log\Big(\frac{\rho_{\rm e}}{\rho_{\rm st}}\Big), \tag{4.5}$$

where $s_{\rm e,st}$ denotes the formation entropy at the standard temperature $T_{\rm st}$ and pressure $p_{\rm st}$, $\rho_{\rm st}$ the mass density at the standard state, and we have $s_{\rm e}(\rho_{\rm e}, T_{\rm e}) = s_{\rm tr}(\rho_{\rm e}, T_{\rm e}) + s_{\rm in}(T_{\rm e})$. It is then obtained that $T_{\rm e}{\rm d}s_{\rm e} = {\rm d}e_{\rm e} - (p_{\rm e}/\rho_{\rm e}^2){\rm d}\rho_{\rm e}$ so that the Gibbs relation at thermodynamic equilibrium may be written

$${\rm d}s_{\rm e} = \frac{c_{\rm ve}}{T_{\rm e}}{\rm d}T_{\rm e} - \frac{r}{\rho_{\rm e}}{\rm d}\rho_{\rm e}. \tag{4.6}$$

From the differential of entropy (4.6), after some algebra, the following governing equation is obtained

$$\partial_t \mathcal{S}_{\rm e} + \boldsymbol{\nabla}\cdot(\boldsymbol{v}_{\rm e}\mathcal{S}_{\rm e}) + \boldsymbol{\nabla}\cdot\Big(\frac{\boldsymbol{Q}_{\rm e}}{T_{\rm e}}\Big) = \mathfrak{v}_{\rm e} = -\frac{\boldsymbol{Q}_{\rm e}\cdot\boldsymbol{\nabla}T_{\rm e}}{T_{\rm e}^2} - \frac{\boldsymbol{\Pi}_{\rm e}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{v}_{\rm e}}{T_{\rm e}}. \tag{4.7}$$

It is easily established that the entropy production—now only due to gradients—is nonnegative. For future use, we introduce the corresponding Gibbs function per unit mass $g_{\rm e} = e_{\rm e} + rT_{\rm e} - T_{\rm e}s_{\rm e}$ as well as the enthalpy per unit mass $h_{\rm e} = e_{\rm e} + rT_{\rm e}$ and we have $g_{\rm e}(\rho_{\rm e}, T_{\rm e}) = g_{\rm tr}(\rho_{\rm e}, T_{\rm e}) + g_{\rm in}(T_{\rm e})$.

The equilibrium viscous tensor is in the form

$$\boldsymbol{\Pi}_{\rm e} = -\kappa_{\rm e}(T_{\rm e})\,(\boldsymbol{\nabla}\cdot\boldsymbol{v}_{\rm e})\boldsymbol{I} - \eta_{\rm e}(T_{\rm e})\,\big(\boldsymbol{\nabla}\boldsymbol{v}_{\rm e} + (\boldsymbol{\nabla}\boldsymbol{v}_{\rm e})^t - \tfrac{2}{d'}(\boldsymbol{\nabla}\cdot\boldsymbol{v}_{\rm e})\boldsymbol{I}\big), \tag{4.8}$$

where $\eta_{\rm e}(T_{\rm e}) = \eta(T_{\rm e}, T_{\rm e})$ and $\kappa_{\rm e}(T_{\rm e}) = \kappa(T_{\rm e}, T_{\rm e})$, and the heat flux is given by

$$\boldsymbol{Q}_{\rm e} = -\lambda_{\rm e}(T_{\rm e})\boldsymbol{\nabla}T_{\rm e}, \tag{4.9}$$

with $\lambda_{\rm e}(T_{\rm e}) = \lambda_{\rm tr,tr}(T_{\rm e},T_{\rm e}) + \lambda_{\rm tr,in}(T_{\rm e},T_{\rm e}) + \lambda_{\rm in,tr}(T_{\rm e},T_{\rm e}) + \lambda_{\rm in,in}(T_{\rm e},T_{\rm e})$.

Letting $n_{\rm e} = d+2$, the conservative variable $\mathsf{u}_{\rm e} \in \mathbb{R}^{n_{\rm e}}$ associated with equations (4.1)–(4.3) is

$$\mathsf{u}_{\rm e} = \big(\rho_{\rm e}, \rho_{\rm e}\boldsymbol{v}_{\rm e}, \mathcal{E}_{\rm e} + \tfrac{1}{2}\rho_{\rm e}\boldsymbol{v}_{\rm e}\cdot\boldsymbol{v}_{\rm e}\big)^t, \tag{4.10}$$

and the corresponding natural variable $\mathsf{z}_{\rm e} \in \mathbb{R}^{n_{\rm e}}$ reads $\mathsf{z}_{\rm e} = \big(\rho_{\rm e}, \boldsymbol{v}_{\rm e}, T_{\rm e}\big)^t$. The corresponding open sets $\mathcal{O}_{\mathsf{u}_{\rm e}}$ and $\mathcal{O}_{\mathsf{z}_{\rm e}}$ of $\mathbb{R}^{n_{\rm e}}$ are given by

$$\mathcal{O}_{\mathsf{u}_{\rm e}} = \big\{\mathsf{u}_{\rm e} = (\mathsf{u}_\rho, \mathsf{u}_{\boldsymbol{v}}, \mathsf{u}_{\rm tl})^t \in \mathbb{R}^{n_{\rm e}};\ \mathsf{u}_\rho > 0,\ \mathsf{u}_{\rm tl} > f^{\rm e}(\mathsf{u}_\rho, \mathsf{u}_{\boldsymbol{v}}\big\}, \tag{4.11}$$

where $f^{\rm e} : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ is given by $f^{\rm e}(\mathsf{u}_\rho, \mathsf{u}_{\boldsymbol{v}}) = \mathsf{u}_\rho e^0_{\rm e} + \frac{1}{2}\frac{\mathsf{u}_{\boldsymbol{v}}\cdot\mathsf{u}_{\boldsymbol{v}}}{\mathsf{u}_\rho}$ and $\mathcal{O}_{\mathsf{z}_{\rm e}} = (0,\infty)\times\mathbb{R}^d\times(0,\infty)$. The map $\mathsf{z}_{\rm e} \to \mathsf{u}_{\rm e}$ is easily shown to be a $C^\varkappa$ diffeomorphism from $\mathcal{O}_{\mathsf{z}_{\rm e}}$ onto $\mathcal{O}_{\mathsf{u}_{\rm e}}$. Introducing the convective and dissipative fluxes of the equilibrium fluid model (4.1)–(4.3)

$$\mathsf{F}^{\rm e}_i = \big(\rho_{\rm e} v_{{\rm e}i},\ \rho_{\rm e}\boldsymbol{v}_{\rm e} v_{{\rm e}i} + p_{\rm e}\boldsymbol{e}_i, (\mathcal{E}_{\rm e} + p_{\rm e} + \tfrac{1}{2}\rho_{\rm e}\boldsymbol{v}_{\rm e}\cdot\boldsymbol{v}_{\rm e})v_{{\rm e}i}\big)^t, \tag{4.12}$$

$$\epsilon_d \mathsf{F}^{\rm e\,diss}_i = \big(0,\ \boldsymbol{\Pi}_{{\rm e}i},\ Q_{{\rm e}i} + \boldsymbol{\Pi}_{{\rm e}i}\cdot\boldsymbol{v}_{\rm e}\big)^t, \tag{4.13}$$

using straightforward notation, the system at equilibrium may be rewritten in qualilinear form

$$\partial_t \mathsf{u}_{\rm e} + \sum_{i\in\mathcal{D}} \mathsf{A}^{\rm e}_i(\mathsf{u}_{\rm e})\partial_i \mathsf{u}_{\rm e} - \epsilon_d \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}^{\rm e}_{ij}(\mathsf{u}_{\rm e})\partial_j \mathsf{u}_{\rm e}\big) = 0, \tag{4.14}$$

where $\mathsf{A}^{\rm e}_i$, $i\in\mathcal{D}$, denote the Jacobian matrices $\mathsf{A}^{\rm e}_i = \partial_{\mathsf{u}_{\rm e}}\mathsf{F}^{\rm e}_i$ and $\mathsf{B}^{\rm e}_{ij}$, $i,j\in\mathcal{D}$, the dissipation matrices at equilibrium with $\mathsf{F}^{\rm e\,diss}_i = -\sum_{j\in\mathcal{D}} \mathsf{B}^{\rm e}_{ij}\partial_j \mathsf{u}_{\rm e}$ [30, 32, 18].

4.2 Symmetrized forms at equilibrium

We evaluate in this section the natural entropic symmetrized form as well as a normal form for the system of partial differential equations modeling fluids at thermodynamic equilibrium (4.1)–(4.3). We use the mathematical entropy $\sigma_{\rm e} = -\mathcal{S}_{\rm e}/r$ where the $1/r$ factor is introduced for convenience. We also relate these symmetric forms to the corresponding forms out of thermodynamic equilibrium.

Theorem 4.1. *Assume that* $(\mathsf{T}_1)(\mathsf{T}_2)$ *and* $(\mathsf{Tr}_1)(\mathsf{Tr}_2)$ *hold. Then the function* $\sigma_{\rm e} = -\mathcal{S}_{\rm e}/r$ *is a mathematical entropy for the system* (4.1)–(4.3) *and the corresponding entropic variable is*

$$\mathsf{v}_{\rm e} = (\partial_{\mathsf{u}_{\rm e}}\sigma_{\rm e})^t = \frac{1}{r}\Big(\frac{g_{\rm e}}{T_{\rm e}} - \frac{\frac{1}{2}|\boldsymbol{v}_{\rm e}|^2}{T_{\rm e}},\ \frac{\boldsymbol{v}_{\rm e}}{T_{\rm e}},\ -\frac{1}{T_{\rm e}}\Big)^t. \tag{4.15}$$

The map $\mathsf{u}_{\rm e} \to \mathsf{v}_{\rm e}$ *is a* $C^{\varkappa-1}$ *diffeomorphism from* $\mathcal{O}_{\mathsf{u}_{\rm e}}$ *given in* (4.11) *onto the open set* $\mathcal{O}_{\mathsf{v}_{\rm e}}$ *given by*

$$\mathcal{O}_{\mathsf{v}_{\rm e}} = \mathbb{R}\times\mathbb{R}^d\times(-\infty,0). \tag{4.16}$$

The system written in terms of the entropic variable $\mathsf{v}_{\rm e}$ *is of the symmetric form*

$$\widetilde{\mathsf{A}}^{\rm e}_0(\mathsf{v}_{\rm e})\partial_t \mathsf{v}_{\rm e} + \sum_{i\in\mathcal{D}} \widetilde{\mathsf{A}}^{\rm e}_i(\mathsf{v}_{\rm e})\partial_i \mathsf{v}_{\rm e} - \epsilon_d \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}^{\rm e}_{ij}(\mathsf{v}_{\rm e})\partial_j \mathsf{v}_{\rm e}\big) = 0, \tag{4.17}$$

where $\widetilde{\mathsf{A}}^{\rm e}_0 = \partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e}$, $\widetilde{\mathsf{A}}^{\rm e}_i = \mathsf{A}^{\rm e}_i\partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e}$, $\widetilde{\mathsf{B}}^{\rm e}_{ij} = \mathsf{B}^{\rm e}_{ij}\partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e}$ *have regularity* $\varkappa - 1$. *The matrix* $\widetilde{\mathsf{A}}^{\rm e}_0$ *is given by*

$$\widetilde{\mathsf{A}}^{\rm e}_0 = \psi^t\,\widetilde{\mathsf{A}}_0(\psi\mathsf{v}_{\rm e})\,\psi = \begin{bmatrix} \rho_{\rm e} & & Sym \\ \rho_{\rm e}\boldsymbol{v}_{\rm e} & \rho_{\rm e}\boldsymbol{v}_{\rm e}\otimes\boldsymbol{v}_{\rm e} + \rho_{\rm e} r T_{\rm e}\boldsymbol{I} & \\ \rho_{\rm e} e_{\rm e,tl} & \rho_{\rm e} h_{\rm e,tl}\boldsymbol{v}^t_{\rm e} & \widetilde{\Upsilon}^{\rm e}_{\rm tl} \end{bmatrix}, \tag{4.18}$$

where $\widetilde{\Upsilon}^{\rm e}_{\rm tl} = \rho_{\rm e}(e_{\rm tl,e})^2 + \rho_{\rm e} r T_{\rm e}(|\boldsymbol{v}_{\rm e}|^2 + c_{\rm v,e}T_{\rm e})$, $e_{\rm e,tl} = e_{\rm e} + \frac{1}{2}|\boldsymbol{v}_{\rm e}|^2$, $h_{\rm e,tl} = h_{\rm e} + \frac{1}{2}|\boldsymbol{v}_{\rm e}|^2$, *and*

$$\psi = \begin{bmatrix} 1 & 0_{1,d} & 0 \\ 0_{d,1} & \boldsymbol{I} & 0_{d,1} \\ 0 & 0_{1,d} & 0 \\ 0 & 0_{1,d} & 1 \end{bmatrix}, \tag{4.19}$$

in such a way that $\psi \mathsf{v}_\mathrm{e}$ is the image of $\mathsf{v}_\mathrm{e} \in \mathbb{R}^{d+2}$ onto the equilibrium manifold $\mathfrak{E}$ in $\mathbb{R}^{d+3}$. Since $\widetilde{\mathsf{A}}_0^\mathrm{e}$ is symmetric, we only give its left lower triangular part and write "Sym" in the upper triangular part. Denoting by $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)^t$ an arbitrary vector of $\mathbb{R}^d$ and letting $\widetilde{\mathsf{A}}^\mathrm{e} = \sum_{i\in\mathcal{D}} \xi_i \widetilde{\mathsf{A}}_i^\mathrm{e}$, we have

$$\widetilde{\mathsf{A}}^\mathrm{e} = \psi^t\, \widetilde{\mathsf{A}}(\psi \mathsf{v}_\mathrm{e})\, \psi = \boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\xi}\, \widetilde{\mathsf{A}}_0^\mathrm{e} + \rho_\mathrm{e} r T_\mathrm{e} \begin{bmatrix} 0 & & \mathit{Sym} \\ \boldsymbol{\xi} & \boldsymbol{\xi}\otimes\boldsymbol{v}_\mathrm{e} + \boldsymbol{v}_\mathrm{e}\otimes\boldsymbol{\xi} & \\ \boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\xi} & \boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\xi}\boldsymbol{v}_\mathrm{e}^t + h_{\mathrm{e,tl}}\boldsymbol{\xi}^t & 2\boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\xi} h_{\mathrm{e,tl}} \end{bmatrix}. \tag{4.20}$$

Moreover, we have the decomposition

$$\widetilde{\mathsf{B}}_{ij}^\mathrm{e} = r\widetilde{\mathsf{B}}^{\lambda,\mathrm{e}}\delta_{ij} + \bar{\kappa}_\mathrm{e} r T_\mathrm{e}\, \widetilde{\mathsf{B}}_{ij}^{\kappa,\mathrm{e}} + \bar{\eta}_\mathrm{e} r T_\mathrm{e}\, \widetilde{\mathsf{B}}_{ij}^{\eta,\mathrm{e}}, \tag{4.21}$$

where

$$\widetilde{\mathsf{B}}^{\lambda,\mathrm{e}} = \psi^t\, \widetilde{\mathsf{B}}^{\lambda}(\psi \mathsf{v}_\mathrm{e})\, \psi = \begin{bmatrix} 0 & & \mathit{Sym} \\ 0_{d,1} & 0_{d,d} & \\ 0 & 0_{1,d} & T_\mathrm{e}^2 \bar{\lambda}_\mathrm{e} \end{bmatrix}, \tag{4.22}$$

and denoting by $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)^t$ and $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_d)^t$ arbitrary vectors of $\mathbb{R}^d$, the matrices $\widetilde{\mathsf{B}}_{ij}^{\kappa,\mathrm{e}}$ and $\widetilde{\mathsf{B}}_{ij}^{\eta,\mathrm{e}}$, $i,j \in \mathcal{D}$, are given by

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \widetilde{\mathsf{B}}_{ij}^{\kappa,\mathrm{e}} = \begin{bmatrix} 0 & 0_d & 0_1 \\ 0_d & \boldsymbol{\xi}\otimes\boldsymbol{\zeta} & \boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\zeta}\,\boldsymbol{\xi} \\ 0_1 & \boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\xi}\,\boldsymbol{\zeta}^t & \boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\xi}\,\boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\zeta} \end{bmatrix}, \tag{4.23}$$

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \widetilde{\mathsf{B}}_{ij}^{\eta,\mathrm{e}} = \begin{bmatrix} 0 & 0_{1,d} & 0 \\ 0_{d,1} & \boldsymbol{\xi}\!\cdot\!\boldsymbol{\zeta}\mathbb{I}_d + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} - \frac{2}{d'}\boldsymbol{\xi}\otimes\boldsymbol{\zeta} & \boldsymbol{\xi}\!\cdot\!\boldsymbol{\zeta}\,\boldsymbol{v}_\mathrm{e} + \boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\xi}\,\boldsymbol{\zeta} - \frac{2}{d'}\boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\zeta}\,\boldsymbol{\xi} \\ 0 & \boldsymbol{\xi}\!\cdot\!\boldsymbol{\zeta}\,\boldsymbol{v}_\mathrm{e}^t + \boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\zeta}\,\boldsymbol{\xi}^t - \frac{2}{d'}\boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\xi}\,\boldsymbol{\zeta}^t & \boldsymbol{\xi}\!\cdot\!\boldsymbol{\zeta}\,\boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{v}_\mathrm{e} + (1-\frac{2}{d'})\boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\xi}\,\boldsymbol{v}_\mathrm{e}\!\cdot\!\boldsymbol{\zeta} \end{bmatrix} \tag{4.24}$$

where $\mathbb{I}$ denotes the identity tensor, and we finally have $\widetilde{\mathsf{B}}_{ij}^{\eta,\mathrm{e}} = \psi^t\, \widetilde{\mathsf{B}}_{ij}^{\eta}(\psi \mathsf{v}_\mathrm{e})\, \psi$ for $i,j\in\mathcal{D}$.

Proof. The explicit calculation of the natural symmetrized system is long but presents no difficulty. The resulting symmetrized form is then seen to be of regularity class $\varkappa - 1$ and is established to correspond with the matrices obtained from the nonequililibrium symmetrized form under the operation $X^\mathrm{e}(\mathsf{v}_\mathrm{e}) = \psi^t X(\psi \mathsf{v}_\mathrm{e})\psi$ where X denotes any of the matrices $\widetilde{\mathsf{A}}_0$, $\widetilde{\mathsf{A}}_i$, $i\in\mathcal{D}$, $\widetilde{\mathsf{B}}^\lambda$, and $\widetilde{\mathsf{B}}_{ij}^\eta$ for $i,j\in\mathcal{D}$. Finally, there are extra new coefficients $\widetilde{\mathsf{B}}_{ij}^{\kappa,\mathrm{e}}$, $i,j\in\mathcal{D}$, associated with the volume viscosity. □

The nullspace invariance property for the symmetrized system (4.17) modeling fluids at thermodynamic equilibrium is easily established with $N(\widetilde{\mathsf{B}}^\mathrm{e}) = \mathbb{R}(1,\mathbf{0},0,)^t$. Since $N(\widetilde{\mathsf{B}}^\mathrm{e})$ is spanned by $(1,\mathbf{0},0,)^t$, we may use the identity matrix $\mathbb{I}_{n_\mathrm{e}}$ as the auxiliary matrix P_e in Theorem 3.7 and select the convenient normal variable

$$\mathsf{w}_\mathrm{e} = \Big(\rho_\mathrm{e}, \boldsymbol{v}_\mathrm{e}, -\frac{1}{T_\mathrm{e}}\Big)^t. \tag{4.25}$$

Theorem 4.2. *Assume that $(\mathsf{T}_1)(\mathsf{T}_2)$ and $(\mathsf{Tr}_1)(\mathsf{Tr}_2)$ hold. Then the map $\mathsf{v}_\mathrm{e} \to \mathsf{w}_\mathrm{e}$ is a $C^{\varkappa-1}$ diffeomorphism from $\mathcal{O}_{\mathsf{v}_\mathrm{e}}$ onto the open set $\mathcal{O}_{\mathsf{w}_\mathrm{e}}$ given by*

$$\mathcal{O}_{\mathsf{w}_\mathrm{e}} = (0,\infty)\times\mathbb{R}^d\times(-\infty,0). \tag{4.26}$$

The system written in the w_e variable is of the normal form

$$\overline{\mathsf{A}}_0^\mathrm{e}(\mathsf{w}_\mathrm{e})\partial_t \mathsf{w}_\mathrm{e} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i^\mathrm{e}(\mathsf{w}_\mathrm{e})\partial_i \mathsf{w}_\mathrm{e} - \epsilon_d \sum_{i,j\in\mathcal{D}} \partial_i\big(\overline{\mathsf{B}}_{ij}^\mathrm{e}(\mathsf{w}_\mathrm{e})\partial_j \mathsf{w}_\mathrm{e}\big) = \epsilon_d \overline{\mathsf{b}}_\mathrm{e}(\mathsf{w}_\mathrm{e}, \partial_\mathsf{x} \mathsf{w}_{\mathrm{IIe}}), \tag{4.27}$$

where $\overline{\mathsf{A}}_0^\mathrm{e} = (\partial_{\mathsf{w}_\mathrm{e}} \mathsf{v}_\mathrm{e})^t\, \widetilde{\mathsf{A}}_0^\mathrm{e}\, \partial_{\mathsf{w}_\mathrm{e}} \mathsf{v}_\mathrm{e}$, $\overline{\mathsf{A}}_i^\mathrm{e}(\partial_{\mathsf{w}_\mathrm{e}} \mathsf{v}_\mathrm{e})^t\, \widetilde{\mathsf{A}}_i^\mathrm{e}\, \partial_{\mathsf{w}_\mathrm{e}} \mathsf{v}_\mathrm{e}$, $\overline{\mathsf{B}}_{ij}^\mathrm{e} = (\partial_{\mathsf{w}_\mathrm{e}} \mathsf{v}_\mathrm{e})^t\, \widetilde{\mathsf{B}}_{ij}^\mathrm{e}\, \partial_{\mathsf{w}_\mathrm{e}} \mathsf{v}_\mathrm{e}$, have regularity $\varkappa - 1$ and $\overline{\mathsf{b}}_\mathrm{e} = -\sum_{i,j\in\mathcal{D}} \partial_i(\partial_{\mathsf{w}_\mathrm{e}} \mathsf{v}_\mathrm{e}^t)\, \widetilde{\mathsf{B}}_{ij}^\mathrm{e}\ \partial_{\mathsf{w}_\mathrm{e}} \mathsf{v}_\mathrm{e}\, \partial_j \mathsf{w}_\mathrm{e}$. The matrix $\overline{\mathsf{A}}_0^\mathrm{e}$ is given by

$$\overline{\mathsf{A}}_0^\mathrm{e} = \psi^t \overline{\mathsf{A}}_0(\psi \mathsf{w}_\mathrm{e})\psi = \begin{bmatrix} \frac{1}{\rho} & & \mathit{Sym} \\ 0_{d,1} & \frac{\rho \boldsymbol{I}}{r T_\mathrm{tr}} & \\ 0 & 0_{1,d} & \frac{\rho_\mathrm{e} T_\mathrm{e}^2 c_{\mathrm{v,e}}}{r} \end{bmatrix},$$

Denoting by $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)^t$ *an arbitrary vector of* $\mathbb{R}^d$*, the matrices* $\overline{\mathsf{A}}_i^{\mathrm{e}}$*,* $i \in \mathcal{D}$*, are given by*

$$\overline{\mathsf{A}}^{\mathrm{e}} = \sum_{i\in\mathcal{D}} \xi_i \overline{\mathsf{A}}_i^{\mathrm{e}} = \psi^t \overline{\mathsf{A}}(\psi \mathsf{w}_{\mathrm{e}})\psi = \overline{\mathsf{A}}_0^{\mathrm{e}} \boldsymbol{v}\cdot\boldsymbol{\xi} + \begin{bmatrix} 0 & & Sym \\ \boldsymbol{\xi} & 0_{d,d} & \\ 0 & \rho_{\mathrm{e}} T_{\mathrm{e}} \boldsymbol{\xi}^t & 0 \end{bmatrix}.$$

The matrices $\overline{\mathsf{B}}_{ij}^{\mathrm{e}}$ *have the structure*

$$\overline{\mathsf{B}}_{ij}^{\mathrm{e}} = \frac{1}{r}\overline{\mathsf{B}}^{\lambda,\mathrm{e}}\delta_{ij} + \frac{\bar{\kappa}}{rT_{\mathrm{e}}}\overline{\mathsf{B}}_{ij}^{\kappa,\mathrm{e}}, + \frac{\bar{\eta}}{rT_{\mathrm{e}}}\overline{\mathsf{B}}_{ij}^{\eta,\mathrm{e}},$$

where the matrix $\overline{\mathsf{B}}^{\lambda,\mathrm{e}}$ *is given by*

$$\overline{\mathsf{B}}^{\lambda,\mathrm{e}} = \psi^t\, \overline{\mathsf{B}}^{\lambda}(\psi \mathsf{w}_{\mathrm{e}})\psi = \begin{bmatrix} 0 & & Sym \\ 0_{d,1} & 0_{d,d} & \\ 0 & 0 & \lambda_{\mathrm{e}} T_{\mathrm{e}}^2 \end{bmatrix}.$$

Denoting by $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)^t$ *and* $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_d)^t$ *arbitrary vectors of* $\mathbb{R}^d$*, the matrices* $\overline{\mathsf{B}}_{ij}^{\eta,\mathrm{e}}$*,* $i, j \in \mathcal{D}$*, are given by*

$$\sum_{i,j\in\mathcal{D}} \xi_i \zeta_j \overline{\mathsf{B}}_{ij}^{\eta,\mathrm{e}} = \begin{bmatrix} 0 & 0_{1,d} & 0 \\ 0_d & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\, \boldsymbol{I} + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} - \frac{2}{d'}\boldsymbol{\xi}\otimes\boldsymbol{\zeta} & 0_{d,1} \\ 0 & 0_{1,d} & 0 \end{bmatrix},$$

and such that $\overline{\mathsf{B}}_{ij}^{\eta,\mathrm{e}} = \psi^t\, \overline{\mathsf{B}}_{ij}^{\eta}(\psi \mathsf{w}_{\mathrm{e}})\psi$ *whereas the matrices* $\overline{\mathsf{B}}_{ij}^{\kappa,\mathrm{e}}$*,* $i, j \in \mathcal{D}$*, are given by*

$$\sum_{i,j\in\mathcal{D}} \xi_i \zeta_j \overline{\mathsf{B}}_{ij}^{\kappa,\mathrm{e}} = \begin{bmatrix} 0 & 0_{1,d} & 0 \\ 0_d & \boldsymbol{\xi}\otimes\boldsymbol{\zeta} & 0_{d,1} \\ 0 & 0_{1,d} & 0 \end{bmatrix}.$$

Finally the quadratic residual $\overline{\mathsf{b}}_{\mathrm{e}}$ *is given by*

$$r\,\overline{\mathsf{b}}_{\mathrm{e}} = \Big(0, \sum_{i\in\mathcal{D}} \boldsymbol{\Pi}_{\mathrm{e}i}\partial_i\Big(\frac{1}{T}\Big), -\boldsymbol{\Pi}_{\mathrm{e}}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{v}_{\mathrm{e}}\Big)^t,$$

and its coefficients have regularity $\varkappa$*.*

Proof. The proof is lengthy and tedious but presents no serious difficulties. □

5 Chapman-Enskog expansion

The Chapman-Enskog asymptotic expansion introduced by Liu [36] and Chen, Levermore and Liu [7] for hyperbolic systems of conservation laws and further investigated by Kawashima and Yong [33] is extended to the situation of hyperbolic-parabolic systems. The resulting first order accurate governing equations for the slow variable then involve dissipative coefficients arising from perturbed convective terms as well as inherited directly from the system out of equilibrium. The expression of dissipative coefficients arising from convective fluxes in the first order accurate equations is also simplified by using convenient generalized inverses with prescribed range and nullspace, their smoothness properties are discussed, as well as symmetrization properties of the resulting system of partial differential equations. It is then established that the viscous fluid equations at equilibrium may be obtained from the Chapman-Enskog expansion in the fast relaxation limit including the volume viscosity term, under the natural scaling $\epsilon = \epsilon_d$. The equilibrium viscous tensor then includes both contributions arising from perturbed convective terms as well as inherited from the non equilibrium fluid and involving respectively the volume and shear viscosities.

5.1 Setting of the problem

An abstract hyperbolic-parabolic system with small second order terms and stiff sources is considered. We assume that the system has an entropy as in Definition 3.1 so that it can be written in symmetric form. We keep the notation of Section 3.1 in order to avoid notational complexities.

We denote by $\mathsf{a}_1, \dots, \mathsf{a}_{n_\mathrm{e}}$ a basis of the slow manifold or equilibrium space $\mathcal{E}$ and by $\mathsf{a}_{n_\mathrm{e}+1}, \dots, \mathsf{a}_n$ a basis of the fast manifold $\mathcal{E}^\perp$. We also introduce the linear operators $\Pi_\mathrm{e} = \mathbb{R}^{n_\mathrm{e}} \longrightarrow \mathbb{R}^n$ and $\Pi_\mathrm{r} = \mathbb{R}^{n-n_\mathrm{e}} \longrightarrow \mathbb{R}^n$ whose matrices in the canonical bases are given by

$$\Pi_\mathrm{e} = [\mathsf{a}_1, \dots, \mathsf{a}_{n_\mathrm{e}}],$$

and

$$\Pi_\mathrm{r} = [\mathsf{a}_{n_\mathrm{e}+1}, \dots, \mathsf{a}_n].$$

We also introduce the metric matrices $\mathcal{J}_\mathrm{e}$ and $\mathcal{J}_\mathrm{r}$ of order n_e and $n - n_\mathrm{e}$, respectively, defined by

$$\mathcal{J}_{\mathrm{e}\,i,j}^{-1} = \langle \mathsf{a}_i, \mathsf{a}_j \rangle, \qquad 1 \le i, j \le n_\mathrm{e},$$

$$\mathcal{J}_{\mathrm{r}\,i,j}^{-1} = \langle \mathsf{a}_i, \mathsf{a}_j \rangle, \qquad n_\mathrm{e} + 1 \le i, j \le n.$$

Each vector $\mathsf{x} \in \mathbb{R}^n$ admits a unique decomposition $\mathsf{x} = \mathsf{x}_\mathcal{E} + \mathsf{x}_{\mathcal{E}^\perp}$ where $\mathsf{x}_\mathcal{E} \in \mathcal{E}$ and $\mathsf{x}_{\mathcal{E}^\perp} \in \mathcal{E}^\perp$. After a little algebra, it is easily shown that, for any $\mathsf{x} \in \mathbb{R}^n$, the vector $\mathcal{J}_\mathrm{e}\Pi_\mathrm{e}^t\mathsf{x}$ represents the coordinates of $\mathsf{x}_\mathcal{E}$ with respect to the vectors $\mathsf{a}_1, \dots, \mathsf{a}_{n_\mathrm{e}}$, whereas the vector $\mathcal{J}_\mathrm{r}\Pi_\mathrm{r}^t\mathsf{x}$ represents the coordinates of $\mathsf{x}_{\mathcal{E}^\perp}$ with respect to the vectors $\mathsf{a}_{n_\mathrm{e}+1}, \dots, \mathsf{a}_n$. One may also easily check that we have the relations

$$\mathsf{x}_\mathcal{E} = \Pi_\mathrm{e}\mathcal{J}_\mathrm{e}\Pi_\mathrm{e}^t\,\mathsf{x}, \qquad \mathsf{x}_{\mathcal{E}^\perp} = \Pi_\mathrm{r}\mathcal{J}_\mathrm{r}\Pi_\mathrm{r}^t\,\mathsf{x},$$

in such a way that we have in $\mathbb{R}^n$

$$\mathbb{I}_n = \Pi_\mathrm{e}\mathcal{J}_\mathrm{e}\Pi_\mathrm{e}^t + \Pi_\mathrm{r}\mathcal{J}_\mathrm{r}\Pi_\mathrm{r}^t,$$

where $\mathbb{I}_n$ denotes the identity tensor in $\mathbb{R}^n$ and, by definition of $\mathcal{J}_\mathrm{e}$ and $\mathcal{J}_\mathrm{r}$, we also have in $\mathbb{R}^{n_\mathrm{e}}$ and in $\mathbb{R}^{n-n_\mathrm{e}}$

$$\mathcal{J}_\mathrm{e}\Pi_\mathrm{e}^t\Pi_\mathrm{e} = \mathbb{I}_{n_\mathrm{e}}, \qquad \mathcal{J}_\mathrm{r}\Pi_\mathrm{r}^t\Pi_\mathrm{r} = \mathbb{I}_{n-n_\mathrm{e}}.$$

We investigate the equilibrium manifold in terms of the symmetric variable v and next in terms of the conservative variable u.

Lemma 5.1. *The equilibrium set in terms of the symmetrizing variable* v *is given by*

$$\{\, \mathsf{v} \in \mathcal{O}_\mathsf{v},\ \ \widetilde{\Omega}(\mathsf{v}) = 0 \,\} = \mathcal{O}_\mathsf{v} \cap \mathcal{E}. \tag{5.1}$$

Proof. This directly results form (S_5). □

This result yields the following properties of the equilibrium set with respect to the conservative variable u.

Lemma 5.2. *The symmetrizing change of variable* $\mathsf{u} \to \mathsf{v}$ *from* $\mathcal{O}_\mathsf{u}$ *onto* $\mathcal{O}_\mathsf{v}$ *induces a diffeomorphism from* $\{\, \mathsf{u} \in \mathcal{O}_\mathsf{u},\ \ \Omega(\mathsf{u}) = 0 \,\}$ *onto* $\mathcal{O}_\mathsf{v} \cap \mathcal{E}$.

Proof. This directly results from $\Omega(\mathsf{u}) = \widetilde{\Omega}(\mathsf{v})$. □

We assume that there exists $\mathsf{u}^\star$ with $\Omega(\mathsf{u}^\star) = 0$ so that the equilibrium manifold $\mathcal{O}_\mathsf{v} \cap \mathcal{E}$ is nontrivial $\mathcal{O}_\mathsf{v} \cap \mathcal{E} \neq \emptyset$. By eventually reducing the open set $\mathcal{O}_\mathsf{u}$ it is then possible to parametrize the equilibrium manifold by its projection $\mathsf{u}_\mathrm{e} = \Pi_\mathrm{e}^t\,\mathsf{u}$ as shown by the following lemma.

Proposition 5.3. *There exists a convex domain* $\mathcal{O}_{\mathsf{u}_\mathrm{e}}$ *containing* $\mathsf{u}_\mathrm{e}^\star = \Pi_\mathrm{e}^t\mathsf{u}^\star$ *such that for any* $\mathsf{u}_\mathrm{e} \in \mathcal{O}_{\mathsf{u}_\mathrm{e}}$ *there exists a unique* $\mathsf{u}_\mathrm{eq} \in \mathcal{O}_\mathsf{u}$ *such that* $\Omega(\mathsf{u}_\mathrm{eq}) = 0$ *and* $\mathsf{u}_\mathrm{e} = \Pi_\mathrm{e}^t\,\mathsf{u}_\mathrm{eq}$. *The map* $\mathsf{u}_\mathrm{e} \to \mathsf{u}_\mathrm{eq}$ *is* $C^\varkappa$ *and its differential satisfies*

$$\partial_\mathsf{u}\Omega\big(\mathsf{u}_\mathrm{eq}\big)\,\partial_{\mathsf{u}_\mathrm{e}}\mathsf{u}_\mathrm{eq} = 0, \qquad \Pi_\mathrm{e}^t\,\partial_{\mathsf{u}_\mathrm{e}}\mathsf{u}_\mathrm{eq} = \mathbb{I}_{n_\mathrm{e}}. \tag{5.2}$$

Denoting by $\mathsf{v}_\mathrm{eq} = \mathsf{v}(\mathsf{u}_\mathrm{eq})$ *the symmetric variable associated with* u_eq, *then the map* $\mathsf{u}_\mathrm{e} \to \mathsf{v}_\mathrm{eq}$ *is at least* $C^{\varkappa-1}$ *and we have* $\mathsf{v}_\mathrm{eq} \in \mathcal{E}$.

Proof. We introduce the map

$$\Phi = (\mathsf{u}, \mathsf{u}_{\rm e})^t \to \big(\Pi_{\rm r}^t \Omega(\mathsf{u}), \Pi_{\rm e}^t \mathsf{u} - \mathsf{u}_{\rm e}\big)^t,$$

from $\mathcal{O}_{\mathsf{u}} \times \Pi_{\rm e}^t \mathcal{O}_{\mathsf{u}}$ to $\mathbb{R}^n$. The map Φ is $C^{\varkappa}$, $\Phi(\mathsf{u}^\star, \mathsf{u}_{\rm e}^\star) = 0$, and we have to establish that its partial differential with respect to u is invertible at $(\mathsf{u}^\star, \mathsf{u}_{\rm e}^\star)^t$. In other words, denoting $(\partial_{\mathsf{u}}\Omega)^\star = \partial_{\mathsf{u}}\Omega(\mathsf{u}^\star)$, we have to establish that the two conditions $\Pi_{\rm r}^t(\partial_{\mathsf{u}}\Omega)^\star \mathsf{x} = 0$ and $\Pi_{\rm e}^t \mathsf{x} = 0$ imply that $\mathsf{x} = 0$. However, letting $\mathsf{x} = \big((\partial_{\mathsf{u}}^2\sigma)^\star\big)^{-1}\mathsf{y}$ the condition $(\partial_{\mathsf{u}}\Omega)^\star \mathsf{x} = (\partial_{\mathsf{v}}\widetilde{\Omega})^\star \mathsf{y} \in \mathcal{E}$ first implies that $(\partial_{\mathsf{v}}\widetilde{\Omega})^\star \mathsf{y} = 0$ since $R\big((\partial_{\mathsf{v}}\widetilde{\Omega})^\star\big) = \mathcal{E}^\perp$ and thus that $\mathsf{y} \in \mathcal{E}$ since $N\big((\partial_{\mathsf{v}}\widetilde{\Omega})^\star\big) = \mathcal{E}$. Using next the condition $\mathsf{x} \in \mathcal{E}^\perp$ we obtain that $\big\langle \big((\partial_{\mathsf{u}}^2\sigma)^\star\big)^{-1}\mathsf{y}, \mathsf{y}\big\rangle = 0$ and thus $\mathsf{y} = 0$, since $\big((\partial_{\mathsf{u}}^2\sigma)^\star\big)^{-1}$ is positive definite, and finally that $\mathsf{x} = 0$. From the implicit function theorem, we may parametrize locally the equilibrium manifold in the form $\big(\mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e}), \mathsf{u}_{\rm e}\big)$ with $\mathsf{u}_{\rm e} \in \mathcal{O}_{\mathsf{u}_{\rm e}}$ and the corresponding map $\mathsf{u}_{\rm e} \to \mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e})$ is $C^{\varkappa}$. Noting that $\Omega = 0$ if and only if $\Pi_{\rm r}^t\Omega = 0$, the differential identities (5.2) are finally obtained by direct differentiation of $\Omega\big(\mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e})\big) = 0$ and $\Pi_{\rm e}^t\big(\mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e})\big) = \mathsf{u}_{\rm e}$. Finally, $\mathsf{u}_{\rm e} \to \mathsf{v}_{\rm eq}$ is at least $C^{\varkappa-1}$ since $\mathsf{u} \to \mathsf{v}$ and $\mathsf{u}_{\rm e} \to \mathsf{u}_{\rm eq}$ are at least $C^{\varkappa-1}$ and we have $\mathsf{v}_{\rm eq} \in \mathcal{E}$ since $\mathsf{u}_{\rm eq}$ is an equilibrium state. □

A first set of governing equations for the variable $\mathsf{u}_{\rm e}$ may next be obtained by applying directly the projection operator $\Pi_{\rm e}^t$ to the governing equations in conservative form and superimposing the equilibrium condition $\mathsf{u} = \mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e})$. These equations are in the form

$$\partial_t \mathsf{u}_{\rm e} + \sum_{i\in\mathcal{D}} \mathsf{A}_i^{\rm e}(\mathsf{u}_{\rm e})\partial_i \mathsf{u}_{\rm e} - \epsilon_d \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}_{ij}^{\rm e}(\mathsf{u}_{\rm e})\partial_j \mathsf{u}_{\rm e}\big) = 0, \tag{5.3}$$

where $\mathsf{A}_i^{\rm e}(\mathsf{u}_{\rm e}) = \Pi_{\rm e}^t \mathsf{A}_i\big(\mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e})\big)\,\partial_{\mathsf{u}_{\rm e}}\mathsf{u}_{\rm eq}$ and $\mathsf{B}_{ij}^{\rm e}(\mathsf{u}_{\rm e}) = \Pi_{\rm e}^t \mathsf{B}_{ij}\big(\mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e})\big)\,\partial_{\mathsf{u}_{\rm e}}\mathsf{u}_{\rm eq}$ where $\mathsf{u}_{\rm eq}$ is the unique equilibrium point obtained in Lemma 5.3.

When ϵ_d remains fixed as $\epsilon \to 0$, this direct method yields the zeroth order governing equations for the variable $\mathsf{u}_{\rm e}$. A typical exemple is for instance the governing equations for chemical equilibrium flows obtained when the chemistry times are faster than the flow times and investigated in [18]. However, *more accurate first order equations* may be obtained by using the Enskog expansion [36, 7] as detailed in the next section in the situation of hyperbolic-parabolic systems. These first order accurate equations are required in particular when the dissipative terms are of first order with respect to ϵ.

5.2 Asymptotic analysis

It is now assumed in this section and in the following that $\epsilon = \epsilon_d$ so that the relaxation time and the dissipative coefficients are of the same order of magnitude. This legitimate scaling is discussed in Appendix A and allows to investigate the double Chapman-Enskog expansion from a kinetic framework to the one-temperature fluid equations.

The solution u of the full system (3.1) is first expanded formally

$$\mathsf{u} = \sum_{i\geq 0} \epsilon^i \overline{\mathsf{u}}_i = \overline{\mathsf{u}}_0 + \epsilon\overline{\mathsf{u}}_1 + \mathcal{O}(\epsilon^2), \tag{5.4}$$

where $\overline{\mathsf{u}}_i$ only depends on $\partial^\alpha \mathsf{u}_{\rm e}$ for $|\alpha| \leq i$. We then write the Enskog constraints

$$\Pi_{\rm e}^t \overline{\mathsf{u}}_0 = \mathsf{u}_{\rm e}, \qquad \Pi_{\rm e}^t \overline{\mathsf{u}}_i = 0, \quad i \geq 1, \tag{5.5}$$

as well as the full equations

$$\begin{aligned} &\partial_t(\overline{\mathsf{u}}_0 + \epsilon\overline{\mathsf{u}}_1 + \cdots) + \sum_{i\in\mathcal{D}} \partial_i\big(\mathsf{F}_i(\overline{\mathsf{u}}_0) + \epsilon\partial_{\mathsf{u}}\mathsf{F}_i(\overline{\mathsf{u}}_0)\overline{\mathsf{u}}_1 + \cdots\big) \\ &\quad - \epsilon \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}_{ij}(\overline{\mathsf{u}}_0)\partial_j\overline{\mathsf{u}}_0 + \cdots\big) = \frac{1}{\epsilon}\big(\Omega(\overline{\mathsf{u}}_0) + \epsilon\partial_{\mathsf{u}}\Omega(\overline{\mathsf{u}}_0)\overline{\mathsf{u}}_1 + \cdots\big). \end{aligned} \tag{5.6}$$

The projected equations then read

$$\partial_t \mathsf{u}_{\rm e} + \sum_{i\in\mathcal{D}} \partial_i\big(\Pi_{\rm e}^t\mathsf{F}_i(\overline{\mathsf{u}}_0) + \epsilon\Pi_{\rm e}^t\partial_{\mathsf{u}}\mathsf{F}_i(\overline{\mathsf{u}}_0)\overline{\mathsf{u}}_1 + \cdots\big) - \epsilon \sum_{i,j\in\mathcal{D}} \partial_i\big(\Pi_{\rm e}^t\mathsf{B}_{ij}(\overline{\mathsf{u}}_0)\partial_j\overline{\mathsf{u}}_0 + \cdots\big) = 0, \tag{5.7}$$

and yield a hierarchy of systems of partial differential equations for the slow variable $\mathsf{u}_{\rm e}$.

The zeroth order approximation is determined from (5.6) by the equilibrium condition

$$\Omega(\overline{\mathsf{u}}_0) = 0,$$

so that $\overline{\mathsf{u}}_0 = \mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e})$ is uniquely defined in terms of $\mathsf{u}_{\rm e}$ from Proposition 5.3, and the zeroth order equations for $\mathsf{u}_{\rm e}$ obtained from (5.7) are in the form

$$\partial_t \mathsf{u}_{\rm e} + \sum_{i\in\mathcal{D}} \partial_i\big(\Pi_{\rm e}^t \mathsf{F}_i(\overline{\mathsf{u}}_0)\big) = 0. \tag{5.8}$$

We will establish later that this yields a symmetrizable hyperbolic system of partial differential equations for the slow variable $\mathsf{u}_{\rm e}$.

At the next order, from (5.5), we obtain for $\overline{\mathsf{u}}_1$ the constraint $\Pi_{\rm e}^t\overline{\mathsf{u}}_1 = 0$, whereas from (5.6), we obtain that

$$\partial_t \overline{\mathsf{u}}_0 + \sum_{i\in\mathcal{D}} \partial_i\big(\mathsf{F}_i(\overline{\mathsf{u}}_0)\big) = \partial_{\mathsf{u}}\Omega(\overline{\mathsf{u}}_0)\overline{\mathsf{u}}_1. \tag{5.9}$$

With the Chapman-Enskog method, the time derivatives in (5.9) are evaluated from the zeroth order equations (5.8) [6, 16, 7, 18, 33] in such a way that

$$\partial_{\mathsf{u}}\Omega(\overline{\mathsf{u}}_0)\overline{\mathsf{u}}_1 = \sum_{i\in\mathcal{D}} \partial_i\big(\mathsf{F}_i(\overline{\mathsf{u}}_0)\big) - \partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0 \sum_{i\in\mathcal{D}} \partial_i\big(\Pi_{\rm e}^t \mathsf{F}_i(\overline{\mathsf{u}}_0)\big). \tag{5.10}$$

We thus have to determine the first order corrector $\overline{\mathsf{u}}_1$ from (5.10) and the constraint $\Pi_{\rm e}^t\overline{\mathsf{u}}_1 = 0$. The first order accurate projected equations are then in the form

$$\partial_t \mathsf{u}_{\rm e} + \sum_{i\in\mathcal{D}} \partial_i\big(\mathsf{F}_i(\overline{\mathsf{u}}_0) + \epsilon\partial_{\mathsf{u}}\mathsf{F}_i(\overline{\mathsf{u}}_0)\overline{\mathsf{u}}_1\big) - \epsilon \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}_{ij}(\overline{\mathsf{u}}_0)\partial_j\overline{\mathsf{u}}_0\big) = 0. \tag{5.11}$$

On the other hand, for $\mathsf{u}_{\rm e} \in \mathcal{O}_{\mathsf{u}_{\rm e}}$ and $\overline{\mathsf{u}}_0 = \overline{\mathsf{u}}_0(\mathsf{u}_{\rm e}) = \mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e})$, denoting by $\widetilde{\mathsf{A}}_0$ the matrix $\widetilde{\mathsf{A}}_0(\overline{\mathsf{u}}_0)$, we have the direct sum

$$\widetilde{\mathsf{A}}_0\mathcal{E} \oplus \mathcal{E}^\perp = \mathbb{R}^n,$$

and since $\big(\widetilde{\mathsf{A}}_0\big)^{-1}\mathcal{E}^\perp = \big(\widetilde{\mathsf{A}}_0\mathcal{E}\big)^\perp$ we have also

$$\big(\widetilde{\mathsf{A}}_0\big)^{-1}\mathcal{E}^\perp \oplus \mathcal{E} = \mathbb{R}^n,$$

and furthermore $N(\partial_{\mathsf{v}}\widetilde{\Omega}) = \mathcal{E}$ and $R(\partial_{\mathsf{v}}\widetilde{\Omega}) = \mathcal{E}^\perp$ since $\overline{\mathsf{u}}_0 = \mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e})$ is an equilibrium state. Using the results from Appendix B, we may introduce the generalized inverse Λ of $-\partial_{\mathsf{v}}\widetilde{\Omega}$ with prescribed range $(\widetilde{\mathsf{A}}_0)^{-1}\mathcal{E}^\perp$ and nullspace $\widetilde{\mathsf{A}}_0\mathcal{E}$. Since $-\partial_{\mathsf{v}}\widetilde{\Omega}$ is symmetric positive semi-definite, we also obtain from Appendix B that Λ is symmetric positive semi-definite. In equation (5.10) we also note that $\partial_{\mathsf{u}}\Omega\,\overline{\mathsf{u}}_1 = \partial_{\mathsf{v}}\widetilde{\Omega}\,(\widetilde{\mathsf{A}}_0)^{-1}\overline{\mathsf{u}}_1$ and that $\partial_i\mathsf{F}_i(\overline{\mathsf{u}}_0) - \partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0\Pi_{\rm e}^t\partial_i\mathsf{F}_i(\overline{\mathsf{u}}_0) \in \mathcal{E}^\perp$ since $\Pi_{\rm e}^t\ \partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0 = \mathbb{I}_{n_{\rm e}}$ and $\Pi_{\rm e}^t\ \big(\partial_i\mathsf{F}_i(\overline{\mathsf{u}}_0) - \partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0\Pi_{\rm e}^t\partial_i\mathsf{F}_i(\overline{\mathsf{u}}_0)\big) = 0$. Noting then that $\mathsf{u}_1 \in \mathcal{E}^\perp$ so that $\big(\widetilde{\mathsf{A}}_0\big)^{-1}\mathsf{u}_1 \in \big(\widetilde{\mathsf{A}}_0\big)^{-1}\mathcal{E}^\perp$, we obtain from the definition of the generalized inverse Λ that

$$(\widetilde{\mathsf{A}}_0)^{-1}\overline{\mathsf{u}}_1 = -\Lambda \sum_{i\in\mathcal{D}}\Big(\partial_i\mathsf{F}_i(\overline{\mathsf{u}}_0) - \partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0\Pi_{\rm e}^t\partial_i\mathsf{F}_i(\overline{\mathsf{u}}_0)\Big).$$

Moreover we note that

$$\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0\partial_i\Pi_{\rm e}^t\mathsf{F}_i(\overline{\mathsf{u}}_0) = \partial_{\mathsf{v}}\mathsf{u}(\overline{\mathsf{u}}_0)\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{v}}_0\partial_i\Pi_{\rm e}^t\mathsf{F}_i(\overline{\mathsf{u}}_0) = \widetilde{\mathsf{A}}_0\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{v}}_0\partial_i\Pi_{\rm e}^t\mathsf{F}_i(\overline{\mathsf{u}}_0),$$

where $\overline{\mathsf{v}}_0$ denotes the symmetrizing variable $\mathsf{v}(\overline{\mathsf{u}}_0)$ corresponding to $\overline{\mathsf{u}}_0$. Moreover $R(\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{v}}_0) \subset \mathcal{E}$ since all $\overline{\mathsf{v}}_0(\mathsf{u}_{\rm e})$ for $\mathsf{u}_{\rm e} \in \mathcal{O}_{\mathsf{u}_{\rm e}}$ are equilibrium states. As a consequence, we have $\partial_{\mathsf{v}}\mathsf{u}(\overline{\mathsf{u}}_0)\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{v}}_0\partial_i\Pi_{\rm e}^t\mathsf{F}_i(\mathsf{u}_0) \in \widetilde{\mathsf{A}}_0\mathcal{E}$ and since $N(\Lambda) = \widetilde{\mathsf{A}}_0\mathcal{E}$, we deduce that

$$\overline{\mathsf{u}}_1 = -\sum_{j\in\mathcal{D}} \widetilde{\mathsf{A}}_0\Lambda\mathsf{A}_j\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0\partial_j\mathsf{u}_{\rm e}.$$

The corrective first order terms in (5.11) may thus be written

$$\begin{aligned}\sum_{i\in\mathcal{D}} \partial_i\big(\Pi_{\mathrm{e}}^t \partial_{\mathsf{u}} \mathsf{F}_i(\overline{\mathsf{u}}_0)\overline{\mathsf{u}}_1\big) &= -\sum_{i,j\in\mathcal{D}} \partial_i\big(\Pi_{\mathrm{e}}^t \mathsf{A}_i \widetilde{\mathsf{A}}_0 \Lambda \mathsf{A}_j \partial_{\mathsf{u}_\mathrm{e}} \overline{\mathsf{u}}_0 \partial_j \mathsf{u}_\mathrm{e}\big)\\ &= -\sum_{i,j\in\mathcal{D}} \partial_i\big(\Pi_{\mathrm{e}}^t \widetilde{\mathsf{A}}_i \Lambda \widetilde{\mathsf{A}}_j \partial_{\mathsf{u}_\mathrm{e}} \overline{\mathsf{v}}_0 \partial_j \mathsf{u}_\mathrm{e}\big),\end{aligned}$$

and we have thus obtained the second order system of partial differential equations

$$\partial_t \mathsf{u}_\mathrm{e} + \sum_{i\in\mathcal{D}} \mathsf{A}_i^\mathrm{e}(\mathsf{u}_\mathrm{e})\partial_i \mathsf{u}_\mathrm{e} - \epsilon \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}_{ij}^\mathrm{e}(\mathsf{u}_\mathrm{e})\partial_j \mathsf{u}_\mathrm{e}\big) = 0, \tag{5.12}$$

where

$$\mathsf{A}_i^\mathrm{e}(\mathsf{u}_\mathrm{e}) = \Pi_\mathrm{e}^t \mathsf{A}_i\, \partial_{\mathsf{u}_\mathrm{e}} \overline{\mathsf{u}}_0,$$
$$\mathsf{B}_{ij}^\mathrm{e}(\mathsf{u}_\mathrm{e}) = \Pi_\mathrm{e}^t \mathsf{B}_{ij}\, \partial_{\mathsf{u}_\mathrm{e}} \overline{\mathsf{u}}_0 + \Pi_\mathrm{e}^t \widetilde{\mathsf{A}}_i \Lambda \widetilde{\mathsf{A}}_j \partial_{\mathsf{u}_\mathrm{e}} \overline{\mathsf{v}}_0.$$

The dissipation matrices $\mathsf{B}_{ij}^\mathrm{e}$ thus include both contributions arising from the perturbed convective terms $\Pi_\mathrm{e}^t \widetilde{\mathsf{A}}_i \Lambda \widetilde{\mathsf{A}}_j \partial_{\mathsf{u}_\mathrm{e}} \overline{\mathsf{v}}_0$ as well as inherited from the original dissipative terms $\Pi_\mathrm{e}^t \mathsf{B}_{ij} \partial_{\mathsf{u}_\mathrm{e}} \overline{\mathsf{u}}_0$. For the application to fluid presented in the next section, for instance, the viscous tensor includes both volume and shear viscosity contributions. We now investigate symmetrizability of the resulting system (5.12) of partial differential equations [7, 33] and simultaneously of both other systems (5.8) and (5.3).

Proposition 5.4. *The $C^\varkappa$ map $\mathsf{u}_\mathrm{e} \to \sigma_\mathrm{e}(\mathsf{u}_\mathrm{e})$ defined over the open convex domain $\mathcal{O}_{\mathsf{u}_\mathrm{e}}$ by*

$$\sigma_\mathrm{e}(\mathsf{u}_\mathrm{e}) = \sigma\big(\mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e})\big),$$

is a mathematical entropy for the system of partial differential equations (5.12) *as well as* (5.8) *and* (5.3)*. Denoting by $\mathsf{v}_\mathrm{eq} = \mathsf{v}(\mathsf{u}_\mathrm{eq})$ the symmetrizing variable corresponding to u_eq, the corresponding entropic variable is given by*

$$\mathsf{v}_\mathrm{e} = \mathcal{J}_\mathrm{e}\Pi_\mathrm{e}^t\, \mathsf{v}_\mathrm{eq},$$

and such that $\mathsf{v}_\mathrm{eq} = \Pi_\mathrm{e}\mathsf{v}_\mathrm{e}$, and the symmetrized equations read

$$\widetilde{\mathsf{A}}_0^\mathrm{e} \partial_t \mathsf{v}_\mathrm{e} + \sum_{i\in\mathcal{D}} \widetilde{\mathsf{A}}_i^\mathrm{e}(\mathsf{v}_\mathrm{e})\partial_i \mathsf{v}_\mathrm{e} - \epsilon \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}_{ij}^\mathrm{e}(\mathsf{v}_\mathrm{e})\partial_j \mathsf{v}_\mathrm{e}\big) = 0, \tag{5.13}$$

where

$$\widetilde{\mathsf{A}}_0^\mathrm{e} = \partial_{\mathsf{v}_\mathrm{e}} \mathsf{u}_\mathrm{e} = \Pi_\mathrm{e}^t \widetilde{\mathsf{A}}_0 \Pi_\mathrm{e}, \qquad \widetilde{\mathsf{A}}_i^\mathrm{e} = \mathsf{A}_i^\mathrm{e} \partial_{\mathsf{v}_\mathrm{e}} \mathsf{u}_\mathrm{e} = \Pi_\mathrm{e}^t \widetilde{\mathsf{A}}_i \Pi_\mathrm{e},$$
$$\widetilde{\mathsf{B}}_{ij}^\mathrm{e} = \mathsf{B}_{ij}^\mathrm{e} \partial_{\mathsf{v}_\mathrm{e}} \mathsf{u}_\mathrm{e} = \Pi_\mathrm{e}^t \widetilde{\mathsf{B}}_{ij} \Pi_\mathrm{e} + \Pi_\mathrm{e}^t \widetilde{\mathsf{A}}_i \Lambda \widetilde{\mathsf{A}}_j \Pi_\mathrm{e},$$

have regularity at least $\varkappa - 2$.

Proof. Keeping in mind that $\overline{\mathsf{u}}_0 = \mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e})$ and $\overline{\mathsf{v}}_0 = \mathsf{v}(\overline{\mathsf{u}}_0)$ we first note that by differentiation that $\mathsf{v}_\mathrm{e}^t = \partial_{\mathsf{u}_\mathrm{e}}\sigma_\mathrm{e} = \partial_{\mathsf{u}}\sigma \partial_{\mathsf{u}_\mathrm{e}} \overline{\mathsf{u}}_0 = \overline{\mathsf{v}}_0^t \partial_{\mathsf{u}_\mathrm{e}} \overline{\mathsf{u}}_0$. However, since $\overline{\mathsf{u}}_0 = \Pi_\mathrm{e}\mathcal{J}_\mathrm{e}\mathsf{u}_\mathrm{e} + \Pi_\mathrm{r}\mathcal{J}_\mathrm{r}\Pi_\mathrm{r}^t\overline{\mathsf{u}}_0$ we have $\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0 = \Pi_\mathrm{e}\mathcal{J}_\mathrm{e} + \Pi_\mathrm{r}\mathcal{J}_\mathrm{r}\Pi_\mathrm{r}^t \partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0$, and furthermore $\overline{\mathsf{v}}_0^t\Pi_\mathrm{r} = 0$ because $\Pi_\mathrm{r}^t\overline{\mathsf{v}}_0 = 0$ at equilibrium. As a consequence, we have $\mathsf{v}_\mathrm{e}^t = \overline{\mathsf{v}}_0^t \Pi_\mathrm{e}\mathcal{J}_\mathrm{e}$ and finally $\mathsf{v}_\mathrm{e} = \mathcal{J}_\mathrm{e}\Pi_\mathrm{e}^t\overline{\mathsf{v}}_0$ keeping in mind that $\mathcal{J}_\mathrm{e}$ is symmetric. This now implies that $\overline{\mathsf{v}}_0 = \Pi_\mathrm{e}\mathsf{v}_\mathrm{e}$ since $\overline{\mathsf{u}}_0$ is an equilibrium point and $\overline{\mathsf{v}}_0 \in \mathcal{E}$.

Differentiating $\mathsf{v}_\mathrm{e} = \mathcal{J}_\mathrm{e}\Pi_\mathrm{e}^t\overline{\mathsf{v}}_0$ with respect to u_e then yields $\partial_{\mathsf{u}_\mathrm{e}}\mathsf{v}_\mathrm{e} = \mathcal{J}_\mathrm{e}\Pi_\mathrm{e}^t\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{v}}_0$ so that

$$\partial_{\mathsf{u}_\mathrm{e}}\mathsf{v}_\mathrm{e} = \mathcal{J}_\mathrm{e}\Pi_\mathrm{e}^t \partial_{\mathsf{u}}\mathsf{v}\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0 = (\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0)^t \partial_{\mathsf{u}}\mathsf{v}(\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0),$$

making use of $\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0 = \Pi_\mathrm{e}\mathcal{J}_\mathrm{e} + \Pi_\mathrm{r}\mathcal{J}_\mathrm{r}\Pi_\mathrm{r}^t\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0$ and $\Pi_\mathrm{r}^t\partial_{\mathsf{u}}\mathsf{v}\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0 = 0$. As a consequence, $\partial_{\mathsf{u}_\mathrm{e}}\mathsf{v}_\mathrm{e} = \partial_{\mathsf{u}_\mathrm{e}}^2\sigma_\mathrm{e}$ is symmetric positive definite as well as $\partial_{\mathsf{v}_\mathrm{e}}\mathsf{u}_\mathrm{e}$ since $\partial_{\mathsf{u}}\mathsf{v}$ is symmetric positive definite.

For the entropy fluxes, using $\partial_{\mathsf{u}_\mathrm{e}}\sigma_\mathrm{e} = \partial_{\mathsf{u}}\sigma\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0$, $\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0 = \Pi_\mathrm{e}\mathcal{J}_\mathrm{e} + \Pi_\mathrm{r}\mathcal{J}_\mathrm{r}\Pi_\mathrm{r}^t\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0$, $\partial_{\mathsf{u}}\sigma(\overline{\mathsf{u}}_0)\Pi_\mathrm{r} = \overline{\mathsf{v}}_0^t\Pi_\mathrm{r} = 0$, and $\mathsf{A}_i^\mathrm{e} = \Pi_\mathrm{e}^t\mathsf{A}_i\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0$, we obtain that

$$\partial_{\mathsf{u}_\mathrm{e}}\sigma_\mathrm{e}\mathsf{A}_i^\mathrm{e} = \overline{\mathsf{v}}_0^t\Pi_\mathrm{e}\mathcal{J}_\mathrm{e}\Pi_\mathrm{e}^t\mathsf{A}_i\partial_{\mathsf{u}_\mathrm{e}}\overline{\mathsf{u}}_0,$$

but since $\overline{\mathsf{v}}_0 \in \mathcal{E}$ so that $\overline{\mathsf{v}}_0 = \Pi_{\rm e}\mathcal{J}_{\rm e}\Pi_{\rm e}^t\overline{\mathsf{v}}_0$ and $\partial_{\mathsf{u}}\sigma\mathsf{A}_i = \partial_{\mathsf{u}}\mathsf{q}$ we obtain that

$$\partial_{\mathsf{u}_{\rm e}}\sigma_{\rm e}\mathsf{A}_i^{\rm e} = \mathsf{v}_0^t\mathsf{A}_i\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0 = \partial_{\mathsf{u}}\mathsf{q}_i\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0 = \partial_{\mathsf{u}_{\rm e}}\mathsf{q}_{ie},$$

where we have defined $\mathsf{q}_{ie} = \mathsf{q}_i\big(\mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e})\big) = \mathsf{q}_i\big(\overline{\mathsf{u}}_0(\mathsf{u}_{\rm e})\big)$.

It is also directly obtained that

$$\widetilde{\mathsf{A}}_i^{\rm e} = \mathsf{A}_i^{\rm e}\partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e} = \Pi_{\rm e}^t\mathsf{A}_i\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0\partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e} = \Pi_{\rm e}^t\mathsf{A}_i\partial_{\mathsf{v}}\mathsf{u}\partial_{\mathsf{v}_{\rm e}}\overline{\mathsf{v}}_0 = \Pi_{\rm e}^t\widetilde{\mathsf{A}}_i\Pi_{\rm e},$$

since $\partial_{\mathsf{v}_{\rm e}}\overline{\mathsf{v}}_0 = \Pi_{\rm e}$, so that $\widetilde{\mathsf{A}}_i^{\rm e}$ is symmetric. Similarly we have

$$\widetilde{\mathsf{A}}_0^{\rm e} = \partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e} = \Pi_{\rm e}^t\partial_{\mathsf{v}_{\rm e}}\overline{\mathsf{u}}_0 = \Pi_{\rm e}^t\partial_{\mathsf{v}}\mathsf{u}\partial_{\mathsf{v}_{\rm e}}\overline{\mathsf{v}}_0 = \Pi_{\rm e}^t\widetilde{\mathsf{A}}_0\Pi_{\rm e}.$$

For the dissipation matrices, we directly obtain that

$$\Pi_{\rm e}^t\mathsf{B}_{ij}\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{u}}_0\partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e} = \Pi_{\rm e}^t\mathsf{B}_{ij}\partial_{\mathsf{v}}\mathsf{u}\partial_{\mathsf{v}_{\rm e}}\overline{\mathsf{v}}_0 = \Pi_{\rm e}^t\widetilde{\mathsf{B}}_{ij}\Pi_{\rm e},$$

and we also have the contributions arising from the convective fluxes

$$\Pi_{\rm e}^t\widetilde{\mathsf{A}}_i\Lambda\widetilde{\mathsf{A}}_j\partial_{\mathsf{u}_{\rm e}}\overline{\mathsf{v}}_0\partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e} = \Pi_{\rm e}^t\widetilde{\mathsf{A}}_i\Lambda\widetilde{\mathsf{A}}_j\partial_{\mathsf{v}_{\rm e}}\overline{\mathsf{v}}_0 = \Pi_{\rm e}^t\widetilde{\mathsf{A}}_i\Lambda\widetilde{\mathsf{A}}_j\Pi_{\rm e},$$

so that finally

$$\widetilde{\mathsf{B}}_{ij}^{\rm e} = \Pi_{\rm e}^t\widetilde{\mathsf{B}}_{ij}\Pi_{\rm e} + \Pi_{\rm e}^t\widetilde{\mathsf{A}}_i\Lambda\widetilde{\mathsf{A}}_j\Pi_{\rm e}.$$

These coefficients satisfy the reciprocity relations $(\widetilde{\mathsf{B}}_{ij}^{\rm e})^t = \widetilde{\mathsf{B}}_{ji}^{\rm e}$ and for any $\boldsymbol{\xi} \in \Sigma^{d-1}$ both matrices $\sum_{i,j\in\mathcal{D}}\xi_i\xi_j\Pi_{\rm e}^t\widetilde{\mathsf{B}}_{ij}^{\rm e}\Pi_{\rm e} = \Pi_{\rm e}^t\widetilde{\mathsf{B}}\Pi_{\rm e}$ and $\sum_{i,j\in\mathcal{D}}\xi_i\xi_j\Pi_{\rm e}^t\widetilde{\mathsf{A}}_i\Lambda\widetilde{\mathsf{A}}_j\Pi_{\rm e} = \Pi_{\rm e}^t\widetilde{\mathsf{A}}\Lambda\widetilde{\mathsf{A}}\Pi_{\rm e}$ are positive semi-definite where $\widetilde{\mathsf{A}} = \sum_{i\in\mathcal{D}}\xi_i\widetilde{\mathsf{A}}_i$. In addition, these coefficients have regularity $\varkappa - 1$ since $\widetilde{\mathsf{B}}_{ij}$, $\widetilde{\mathsf{A}}_i$, and Λ have regularity $\varkappa - 1$ keeping in mind that $\widetilde{\mathsf{A}}_0$ and $\partial_{\mathsf{v}}\widetilde{\Omega}$ have regularity $\varkappa - 1$ and using the smoothness of Λ in terms of $\widetilde{\mathsf{A}}_0$ and $\partial_{\mathsf{v}}\widetilde{\Omega}$ deduced from (B.1). □

Remark 5.5. *In this section, the natural outer expansion for solutions of the system of partial differential equations* (3.1) *have been investigated. Such solutions, however, cannot have inital values out of equilibrium and need to be completed with initial layer correctors. These initial correctors may generally investigated by using composite expansions [48, 35] in the form*

$$\mathsf{u} = \sum_{i\geq 0}\epsilon^i\big(\overline{\mathsf{u}}_i(t,x) + \mathsf{u}_i^l(\tau,x)\big), \tag{5.14}$$

where $\sum_{i\geq 0}\epsilon^i\overline{\mathsf{u}}_i$ denotes the outer Chapman-Enskog expansion and $\sum_{i\geq 0}\epsilon^i\mathsf{u}_i^l$ the inner expansion with $\tau = t/\epsilon$. The initial layer correctors u_i^l must decrease exponentially to zero as $\tau\to\infty$ and are only significant for small times since $\tau = t/\epsilon$. The analysis of these composite expansion lay outside the range of the present work.

5.3 Application to internal energy relaxation

We apply in this section the Chapman-Enskog method to the situation of gases out of thermodynamic equilibrium with fast internal energy relaxation limit. In this situation, the conservative variable is given by

$$\mathsf{u} = \big(\rho, \rho\boldsymbol{v}, \mathcal{E}_{\rm in}, \mathcal{E}_{\rm tr} + \mathcal{E}_{\rm in} + \tfrac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v}\big)^t,$$

and the corresponding conservative variable at equilibrium is given by

$$\mathsf{u}_{\rm e} = \big(\rho, \rho\boldsymbol{v}, \mathcal{E} + \tfrac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v}\big)^t.$$

The equilibrium manifold $\mathcal{E}$ is given by

$$\mathcal{E} = \mathrm{span}\{\mathsf{e}_1, \mathsf{e}_{1+1}, \dots, \mathsf{e}_{1+d}, \mathsf{e}_{3+d}\}, \qquad \mathcal{E}^\perp = \mathbb{R}\mathsf{e}_{2+d},$$

where $\mathsf{e}_1,\dots,\mathsf{e}_n$ denote the basis vectors of $\mathbb{R}^n$, and the corresponding projector operators are given by

$$\psi = \Pi_{\mathrm{e}} = \begin{bmatrix} 1 & 0_{1,d} & 0 \\ 0_{d,1} & \boldsymbol{I} & 0_{d,1} \\ 0 & 0_{1,d} & 0 \\ 0 & 0_{1,d} & 1 \end{bmatrix}, \qquad \Pi_{\mathrm{r}} = \begin{bmatrix} 0 \\ 0_{d,1} \\ 1 \\ 0 \end{bmatrix},$$

and $\mathcal{J}_{\mathrm{e}} = \mathbb{I}_{2+d}$, $\mathcal{J}_{\mathrm{r}} = 1$. In order to evaluate the Jacobian matrices at equilibrium, it is then practical to use the natural variables

$$\mathsf{z} = \big(\rho, \boldsymbol{v}, T_{\mathrm{in}}, T_{\mathrm{tr}}\big)^t, \qquad \mathsf{z}_{\mathrm{e}} = \big(\rho, \boldsymbol{v}, T\big)^t,$$

and at equilibrium we have $T_{\mathrm{tr}} = T_{\mathrm{in}} = T$ so that $\mathsf{z}_{\mathrm{eq}} = \overline{\mathsf{z}}_0 = \big(\rho, \boldsymbol{v}, T, T\big)^t$, $\overline{\mathsf{u}}_0 = \mathsf{u}(\overline{\mathsf{z}}_0)$ and

$$\partial_{\mathsf{z}_{\mathrm{e}}} \overline{\mathsf{z}}_0 = \begin{bmatrix} 1 & 0_{1,d} & 0 \\ 0_{d,1} & \boldsymbol{I} & 0_{d,1} \\ 0 & 0_{1,d} & 1 \\ 0 & 0_{1,d} & 1 \end{bmatrix}.$$

After some algebra, we may evaluate

$$\partial_{\mathsf{u}_{\mathrm{e}}} \overline{\mathsf{u}}_0 = \partial_{\mathsf{z}} \mathsf{u}(\overline{\mathsf{z}}_0)\, \partial_{\mathsf{z}_{\mathrm{e}}} \overline{\mathsf{z}}_0\, \partial_{\mathsf{u}_{\mathrm{e}}} \mathsf{z}_{\mathrm{e}} = \partial_{\mathsf{z}} \mathsf{u}(\overline{\mathsf{z}}_0)\, \partial_{\mathsf{z}_{\mathrm{e}}} \overline{\mathsf{z}}_0\, (\partial_{\mathsf{z}_{\mathrm{e}}} \mathsf{u}_{\mathrm{e}})^{-1},$$

on the equilibrium manifold

$$\partial_{\mathsf{u}_{\mathrm{e}}} \overline{\mathsf{u}}_0 = \begin{bmatrix} 1 & 0_{1,d} & 0 \\ 0_{d,1} & \boldsymbol{I} & 0_{d,1} \\ \alpha_\rho & \alpha_{\boldsymbol{v}} & \alpha_T \\ 0 & 0_{1,d} & 1 \end{bmatrix},$$

where

$$\alpha_\rho = \frac{1}{c_{\mathrm{v}}}\big(c_{\mathrm{tr}} e_{\mathrm{in}} - c_{\mathrm{in}} e_{\mathrm{tr}} + c_{\mathrm{in}} \tfrac{1}{2} |\boldsymbol{v}|^2\big), \qquad \alpha_{\boldsymbol{v}} = -\frac{c_{\mathrm{in}}}{c_{\mathrm{v}}} \boldsymbol{v}^t, \qquad \alpha_T = \frac{c_{\mathrm{in}}}{c_{\mathrm{v}}}.$$

We next evaluate the default fluxes $\sum_{i\in\mathcal{D}} \partial_i\big(\mathsf{F}_i(\overline{\mathsf{u}}_0)\big) - \partial_{\mathsf{u}_{\mathrm{e}}} \overline{\mathsf{u}}_0 \sum_{i\in\mathcal{D}} \partial_i\big(\Pi_{\mathrm{e}}^t \mathsf{F}_i(\overline{\mathsf{u}}_0)\big)$ with

$$\mathsf{F}_i = \big(\rho v_i, \rho \boldsymbol{v} v_i + p \boldsymbol{e}_i, \mathcal{E}_{\mathrm{in}} v_i, (\mathcal{E}_{\mathrm{tr}} + \mathcal{E}_{\mathrm{in}} + p + \tfrac{1}{2} \rho \boldsymbol{v}\cdot\boldsymbol{v}) v_i\big)^t,$$

$$\mathsf{F}_i^{\mathrm{e}} = \big(\rho v_i, \rho \boldsymbol{v} v_i + p \boldsymbol{e}_i, (\mathcal{E} + p + \tfrac{1}{2} \rho \boldsymbol{v}\cdot\boldsymbol{v}) v_i\big)^t,$$

using $T_{\mathrm{tr}} = T_{\mathrm{in}} = T$ and after some lengthy algebra we obtain

$$\sum_{i\in\mathcal{D}} \partial_i\big(\mathsf{F}_i(\overline{\mathsf{u}}_0)\big) - \partial_{\mathsf{u}_{\mathrm{e}}} \overline{\mathsf{u}}_0 \sum_{i\in\mathcal{D}} \partial_i\big(\Pi_{\mathrm{e}}^t \mathsf{F}_i(\overline{\mathsf{u}}_0)\big) = \Big(0, \boldsymbol{0}, -\frac{c_{\mathrm{in}}}{c_{\mathrm{v}}} p \boldsymbol{\nabla}\cdot\boldsymbol{v}, 0\Big)^t.$$

In order to evaluate the corrector $\overline{\mathsf{u}}_1$, we need to evaluate the Jacobian of the source term at equilibrium $\partial_{\mathsf{u}}\Omega(\overline{\mathsf{u}}_0)$. Such a calculation is again conveniently done by using the natural variable since $\partial_{\mathsf{u}}\Omega = \partial_{\mathsf{z}}\Omega(\partial_{\mathsf{z}}\mathsf{u})^{-1}$ and after some algebra we obtain that

$$\partial_{\mathsf{u}}\Omega(\overline{\mathsf{u}}_0) = \frac{1}{c_{\mathrm{tr}} \overline{\tau}_{\mathrm{in}}} \begin{bmatrix} 0 & 0_{1,d} & 0 & 0 \\ 0_{d,1} & 0_{d,d} & 0_{d,1} & 0_{d,1} \\ \gamma & -c_{\mathrm{in}} \boldsymbol{v}^t & -c_{\mathrm{v}} & c_{\mathrm{in}} \\ 0 & 0_{1,d} & 0 & 0 \end{bmatrix},$$

where $\gamma = c_{\rm tr} e_{\rm in} - c_{\rm in} e_{\rm tr} + c_{\rm in}\frac{1}{2}|\boldsymbol{v}|^2$. The constraint $\Pi_{\rm e}^t \overline{\mathsf{u}}_1 = 0$ then yields that $\overline{\mathsf{u}}_1 = \big(0, 0_{d,1}, \overline{\mathsf{u}}_{1\rm in}, 0\big)^t$ and from the equation for $\overline{\mathsf{u}}_1$ we finally obtain that

$$\overline{\mathsf{u}}_1 = \overline{\mathsf{u}}_{1\rm in}\ \mathsf{e}_{2+d} = \frac{c_{\rm tr} c_{\rm in}}{c_{\rm v}^2} \bar{\tau}_{\rm in} p \boldsymbol{\nabla}\cdot\boldsymbol{v}\ \mathsf{e}_{2+d}.$$

Defining the rescaled equilibrium volume viscosity as

$$\bar{\kappa} = \frac{r\, c_{\rm in}\, p \bar{\tau}_{\rm in}}{c_{\rm v}^2}, \tag{5.15}$$

where all terms are evaluated at z_0, the first order temperature difference $(T_{\rm tr} - T)_1$ is found to be after some algebra $\rho r (T_{\rm tr} - T)_1 = -\bar{\kappa} \boldsymbol{\nabla}\cdot\boldsymbol{v}$ which corresponds to the classical temperature estimate rewritten in the rigorous asymptotic framework.

Finally, the corrective term in the governing equations $\sum_{i\in\mathcal{D}} \partial_i \big(\Pi_{\rm e}^t \partial_{\mathsf{u}} \mathsf{F}_i \overline{\mathsf{u}}_1\big)$ is evaluated in the form $\sum_{i\in\mathcal{D}} \partial_i \big(\Pi_{\rm e}^t \partial_{\mathsf{z}} \mathsf{F}_i (\partial_{\mathsf{z}} \mathsf{u})^{-1} \overline{\mathsf{u}}_1\big)$ and is found to be

$$\sum_{i\in\mathcal{D}} \partial_i \big(\Pi_{\rm e}^t \partial_{\mathsf{u}} \mathsf{F}_i \overline{\mathsf{u}}_1\big) = -\big(0, \boldsymbol{\nabla}\cdot(\bar{\kappa}\boldsymbol{\nabla}\cdot\boldsymbol{v}\boldsymbol{I}), \boldsymbol{\nabla}\cdot(\bar{\kappa}\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\boldsymbol{v})\big)^t, \tag{5.16}$$

and its components are precisely the missing volume viscosity terms in the slow variable governing equations. As a consequence, the first order equations exactly yields the viscous Navier-Stokes equations *with* the volume viscosity terms (5.16) and with the traditional relation between volume viscosity and the relaxation time for internal energy (5.15). The dissipative terms include both contributions from the internal energy relaxation with the volume viscosity as well as inherited from the original system with the shear viscosity and the thermal conductivity. Moreover, the explicit symmetrized form obtained with Theorem 4.1 corresponds to the general symmetrizing result of Proposition 5.4 with the identities $\Lambda = (pTc_{\rm tr}^2 c_{\rm in}\bar{\tau}_{\rm in}/c_{\rm v}^2)\ (\widetilde{\mathsf{A}}_0)^{-1}\mathsf{e}_{d+2} \otimes (\widetilde{\mathsf{A}}_0)^{-1}\mathsf{e}_{d+2}$ and $\bar{\kappa} r T\ \widetilde{\mathsf{B}}_{ij}^{\kappa} = \Pi_{\rm e}^t \widetilde{\mathsf{A}}_i \Lambda \widetilde{\mathsf{A}}_i \Pi_{\rm e}$.

5.4 Double Chapman-Enskog expansions

A consequence of the previous section is that the one-temperature Navier-Stokes equations may be obtained by using two different procedures. A first one consists in deriving the out of equilibrium fluid model from the kinetic framework by a first Chapman-Enskog method with $\epsilon_d \to 0$ and ϵ finite, and then to perform a second Chapman-Enskog method with $\epsilon = \epsilon_d \to 0$ starting from the out of equilibrium fluid. The second procedure consists in deriving directly the one-temperature model from the kinetic model by a single Chapman-Enskog method with $\epsilon = \epsilon_d \to 0$ as usual [6, 16, 18]. These two different paths are illustrated in Figure 1 and it is then legitimate to compare the resulting mathematical models. This problem is similar to the situation of chemical equilibrium fluids which may either be derived directly from a kinetic model [14] or else be obtained by superimposing chemical equilibrium in reactive fluid equations [18].

$$\begin{tikzcd} \text{Kinetic} \arrow[r, "\epsilon_d \ll 1"] \arrow[d, "\epsilon_d = \epsilon \ll 1"'] & \text{Nonequilibrium Fluid} \arrow[dl, "\epsilon_d = \epsilon \ll 1"] \\ \text{Equilibrium Fluid} & \end{tikzcd}$$

Figure 1: Schematic of the double Chapman-Enskog procedure

On the first hand, there are very strong similarities between the resulting systems of differential equations. The same conservation equations, transport fluxes and thermodynamic properties are indeed obtained, so that the systems coincide at zeroth order in particular. The mathematical properties of the transport coefficients are also similar as well as the general mathematical structure and symmetrizing properties.

On the other hand, there are differences in the quantitative values of the transport coefficients. Indeed, in principle, the full linearized collision operator plays a role in the transport linear systems of the equilibrium model obtained directly, whereas only the linearized fast collision operator plays a role for the double Chapman-Enskog procedure. For equilibrium chemistry flows for instance the linearized reactive collision operator is not taken into account in the diffusion coefficients associated with the double Chapman-Enskog procedure although it does play a role in the kinetic equilibrium regime derivation [14]. These effects are nevertheless very small and, in practice, the system obtained from the double Chapman-Enskog procedure is used for chemical equilibrium flows [18] as well as partial equilibrium flows [20]. In the special situation of nonequilibrium gases, these quantitative differences are also negligible, since for λ we recover the nonelastic contributions by summing $\lambda = \lambda_{\mathrm{tr,tr}} + \lambda_{\mathrm{tr,in}} + \lambda_{\mathrm{in,tr}} + \lambda_{\mathrm{in,in}}$ whereas the internal degrees of freedom generally have a weak influence on the shear viscosity.

6 Conclusion

The equations governing fluids out of equilibrium—derived from the kinetic theory of gases—have been presented. The corresponding quasilinear system of partial differential equations has been recast in natural symmetric form as well in normal form. This normal form has important properties that will be critical for a priori estimates and existence results [24], namely the commutation between the mass matrix $\overline{\mathsf{A}}_0$ and the orthogonal projector π onto the equilibrium manifold $\overline{\mathcal{E}}^{\perp}$.

The Chapman-Enskog expansion has been generalized to the situation of hyperbolic-parabolic systems. Symmetrization of the corresponding systems has been investigated as well as simplifications for the expression of the extra dissipative coefficients arising from the perturbed convective terms and their smoothness properties. Finally, applying these abstract results to the situation of nonequilibrium gases, we have established that volume viscosity terms are obtained as first order perturbations in the Chapman-Enskog expansion thereby providing a general rigorous framework for the fast internal relaxation limit more satisfactory than traditional physically intuitive methods.

A A Two-Temperature Kinetic Framework

We summarize a nonequilibrium kinetic model for a polyatomic gas and the derivation of the corresponding fluid governing equations from the Chapman-Enskog method [3, 4]. In a kinetic framework for a single polyatomic gas, the Boltzmann equation may be written in the form

$$\partial_t f + \boldsymbol{c}\cdot\boldsymbol{\nabla} f = \frac{1}{\epsilon_d}\mathcal{J}^{\mathrm{f}} + \mathcal{J}^{\mathrm{s}}, \tag{A.1}$$

where t denotes time, ∂_t the time derivative operator, $\boldsymbol{x}$ the spatial coordinate, ∇ the space derivative operator, $\boldsymbol{c}$ the particle velocity, $f(t,\boldsymbol{x},\boldsymbol{c},\mathrm{I})$ the distribution function, I the index of the quantum energy state, $\mathcal{J}^{\mathrm{f}}$ the fast collision operator, $\mathcal{J}^{\mathrm{s}}$ the slow collision operator, and ϵ_d a typical Knudsen number.

The complete collision operator $\mathcal{J} = \mathcal{J}^{\mathrm{f}} + \mathcal{J}^{\mathrm{s}}$ is in the form

$$\mathcal{J}(f) = \sum_{\mathrm{J},\mathrm{I}',\mathrm{J}'} \int \Big(f(\boldsymbol{c}',\mathrm{I}')f(\widetilde{\boldsymbol{c}}',\mathrm{J}')\frac{a_{\mathrm{I}}a_{\mathrm{J}}}{a_{\mathrm{I}'}a_{\mathrm{J}'}} - f(\boldsymbol{c},\mathrm{I})f(\widetilde{\boldsymbol{c}},\mathrm{J})\Big) g\sigma^{\mathrm{IJI'J'}}\,\mathrm{d}\widetilde{\boldsymbol{c}}\,\mathrm{d}\boldsymbol{e}', \tag{A.2}$$

where I and J denote the indices of the quantum energy states before collision, I' and J' the corresponding numbers after collision, $\widetilde{\boldsymbol{c}}$ the velocity of the colliding partner, $\boldsymbol{c}'$ and $\widetilde{\boldsymbol{c}}'$ the velocities after collision, a_{I} the degeneracy of the Ith quantum energy state, $\sigma^{\mathrm{IJI'J'}}$ the collision cross section, g the absolute value of the relative velocity $\boldsymbol{c}-\widetilde{\boldsymbol{c}}$ of the incoming particles and $\boldsymbol{e}'$ the unit vector in the direction of the relative velocity $\boldsymbol{c}'-\widetilde{\boldsymbol{c}}'$ after collision. Only binary collisions are considered since the system is dilute and the cross sections satisfy the reciprocity relations [46, 18]

$$a_{\mathrm{I}}a_{\mathrm{J}}g\sigma^{\mathrm{IJI'J'}}\mathrm{d}\boldsymbol{c}\,\mathrm{d}\widetilde{\boldsymbol{c}}\,\mathrm{d}\boldsymbol{e}' = a_{\mathrm{I}}'a_{\mathrm{J}}'g'\sigma^{\mathrm{I'J'IJ}}\mathrm{d}\boldsymbol{c}'\mathrm{d}\widetilde{\boldsymbol{c}}'\mathrm{d}\boldsymbol{e}. \tag{A.3}$$

Denoting by E_{I} the internal energy of the particle in the Ith state, we write $\Delta E = E_{\mathrm{I}'} + E_{\mathrm{J}'} - E_{\mathrm{I}} - E_{\mathrm{J}}$ for the energy jump. The fast collision operator $\mathcal{J}^{\mathrm{f}}$ includes all collisions satisfying $\Delta E = 0$, either involving only the translational energies or resonant with respect to the internal energy, and the slow collision operator $\mathcal{J}^{\mathrm{s}}$ describes the collisions for which $\Delta E \neq 0$. Assuming that there are sufficiently resonant collisions, and denoting by $\mathfrak{m}$ the mass per molecule, the collisional invariants of the fast collision operator are associated with particle number $\psi^1 = 1$, momentum $\psi^{1+l} = \mathfrak{m}c_l$, $l \in \{1,2,3\}$, the energy associated with translational degrees of freedom $\psi^5 = \psi_{\mathrm{tr}}$ and the internal energy mode $\psi^6 = \psi_{\mathrm{in}}$, where $\psi_{\mathrm{tr}} = \frac{1}{2}\mathfrak{m}(\boldsymbol{c}-\boldsymbol{v})\cdot(\boldsymbol{c}-\boldsymbol{v})$ and $\psi_{\mathrm{in}} = E_{\mathrm{I}}$.

The Enskog expansion is in the form $f = f^{(0)}\big(1 + \epsilon_d \phi + \mathcal{O}(\epsilon_d^2)\big)$ where $f^{(0)}$ is the Maxwellian distribution and ϕ the perturbation associated with the Navier-Stokes regime. This Maxwellian distribution involves two temperatures and is in the form

$$f^{(0)} = \Big(\frac{\mathfrak{m}}{2\pi k_{\mathrm{B}} T_{\mathrm{tr}}}\Big)^{\frac{3}{2}} \frac{\mathfrak{n} a_{\mathrm{I}}}{Z_{\mathrm{in}}} \exp\Big(-\frac{\mathfrak{m}(\boldsymbol{c}-\boldsymbol{v})\cdot(\boldsymbol{c}-\boldsymbol{v})}{2k_{\mathrm{B}}T_{\mathrm{tr}}} - \frac{E_{\mathrm{I}}}{k_{\mathrm{B}}T_{\mathrm{in}}}\Big), \tag{A.4}$$

where $\mathfrak{n}$ is the number of molecules per unit volume, T_{tr} the translational temperature, T_{in} the temperature associated with the internal energy modes, and $Z_{\mathrm{in}} = \sum_{\mathrm{I}} a_{\mathrm{I}} \exp\big(-E_{\mathrm{I}}/k_{\mathrm{B}}T_{\mathrm{in}}\big)$ the internal partition function which only depends on T_{in}.

The equations for conservation of mass, momentum and internal energies are obtained by taking the scalar product of the Boltzmann equation (A.1) with the collisional invariants of the fast collision operator. The scalar product $\langle\!\langle \xi, \zeta \rangle\!\rangle$ between two tensorial quantities $\xi(t,\boldsymbol{x},\boldsymbol{c},\mathrm{I})$ and $\zeta(t,\boldsymbol{x},\boldsymbol{c},\mathrm{I})$ is defined by $\langle\!\langle \xi, \zeta \rangle\!\rangle = \sum_{\mathrm{I}} \int \xi \odot \zeta \, \mathrm{d}\boldsymbol{c}$ where $\xi \odot \zeta$ is the contracted product. The fluid variables are the particle number density $\mathfrak{n} = \langle\!\langle \psi^1, f \rangle\!\rangle = \langle\!\langle \psi^1, f^{(0)} \rangle\!\rangle$ or equivalently the mass density $\rho = \mathfrak{m}\mathfrak{n}$, the mass averaged velocity $\boldsymbol{v}$ such that $\rho\boldsymbol{v} = \langle\!\langle \mathfrak{m}\boldsymbol{c}, f \rangle\!\rangle = \langle\!\langle \mathfrak{m}\boldsymbol{c}, f^{(0)} \rangle\!\rangle$, and the translation and internal temperatures T_{tr} and T_{in} defined by $\mathcal{E}_{\mathrm{tr}}(\rho, T_{\mathrm{tr}}) = \langle\!\langle f, \psi_{\mathrm{tr}} \rangle\!\rangle = \langle\!\langle f^{(0)}, \psi_{\mathrm{tr}} \rangle\!\rangle$ and $\mathcal{E}_{\mathrm{in}}(\rho, T_{\mathrm{in}}) = \langle\!\langle f, \psi_{\mathrm{in}} \rangle\!\rangle = \langle\!\langle f^{(0)}, \psi_{\mathrm{in}} \rangle\!\rangle$ where $\mathcal{E}_{\mathrm{tr}}$ and $\mathcal{E}_{\mathrm{in}}$ denote the internal energies per unit volume of translational and internal origin, respectively. Following the Chapman-Enskog procedure of the kinetic theory of gases, the equations for conservation of mass (2.1), momentum (2.2) and internal energy (2.3) and total energy (2.4) are then established as well as the expressions for heat fluxes (2.14) (2.15) and the viscous tensor (2.16), and the thermodynamic properties (2.6)–(2.9) presented in Section 2 [38, 3, 4].

The fluid entropy per unit volume reads $\mathcal{S} = -k_{\mathrm{B}} \sum_{\mathrm{I}} \int f^{(0)}\big(\log(f^{(0)}\beta_{\mathrm{I}}) - 1\big)\mathrm{d}\boldsymbol{c}$ where $\beta_{\mathrm{I}} = h_{\mathrm{P}}^3/(a_{\mathrm{I}} m^3)$ and h_{P} is the Planck constant. After some algebra, $\mathcal{S}$ is found in the form $\mathcal{S} = \mathcal{S}_{\mathrm{tr}} + \mathcal{S}_{\mathrm{in}}$ where the translational entropy per unit volume $\mathcal{S}_{\mathrm{tr}}$ and the internal entropy per unit volume $\mathcal{S}_{\mathrm{in}}$ read

$$\mathcal{S}_{\mathrm{tr}} = \mathfrak{n} k_{\mathrm{B}} \Big(\frac{5}{2} - \log\frac{\mathfrak{n}}{Z_{\mathrm{tr}}}\Big), \qquad \mathcal{S}_{\mathrm{in}} = \mathfrak{n} k_{\mathrm{B}} \Big(\frac{\overline{E}}{k_{\mathrm{B}}T_{\mathrm{in}}} - \log\frac{1}{Z_{\mathrm{in}}}\Big), \tag{A.5}$$

where $Z_{\mathrm{tr}} = \big(\frac{2\pi \mathfrak{m} k_{\mathrm{B}} T_{\mathrm{tr}}}{h_{\mathrm{P}}^2}\big)^{3/2}$ denotes the translational partition function per unit volume and $\overline{E} = (1/Z_{\mathrm{in}}) \sum_{\mathrm{I}} a_{\mathrm{I}} E_{\mathrm{I}} \exp\big(-E_{\mathrm{I}}/k_{\mathrm{B}}T_{\mathrm{in}}\big)$ the average energy. Defining the translational and internal Gibbs functions per particle $G_{\mathrm{tr}} = k_{\mathrm{B}} T_{\mathrm{tr}} \log\frac{\mathfrak{n}}{Z_{\mathrm{tr}}}$ and $G_{\mathrm{in}} = k_{\mathrm{B}} T_{\mathrm{in}} \log\frac{1}{Z_{\mathrm{in}}}$, the translational and internal Gibbs' relations are in the form $T_{\mathrm{tr}}\mathrm{d}\mathcal{S}_{\mathrm{tr}} = \mathrm{d}\mathcal{E}_{\mathrm{tr}} - G_{\mathrm{tr}}\mathrm{d}\mathfrak{n}$ and $T_{\mathrm{in}}\mathrm{d}\mathcal{S}_{\mathrm{in}} = \mathrm{d}\mathcal{E}_{\mathrm{in}} - G_{\mathrm{in}}\mathrm{d}\mathfrak{n}$ and the full entropy differential is easily obtained.

A direct evaluation of the source term ω_{in} also yields that

$$\omega_{\mathrm{in}} = -2\mathfrak{n}^2 \Big[\!\!\Big[(\Delta E)\Big(\exp\big(\tfrac{\Delta E}{k_{\mathrm{B}}T_{\mathrm{tr}}} - \tfrac{\Delta E}{k_{\mathrm{B}}T_{\mathrm{in}}}\big) - 1\Big) \Big]\!\!\Big], \tag{A.6}$$

where $[\![\alpha]\!] = \frac{1}{8\mathfrak{n}^2} \sum_{\mathrm{I,J,I',J'}} \int \alpha_{\mathrm{IJI'J'}} f^{(0)} \widetilde{f}^{(0)} g \sigma^{\mathrm{IJI'J'}} \mathrm{d}\boldsymbol{c}\, \mathrm{d}\widetilde{\boldsymbol{c}}\, \mathrm{d}\boldsymbol{e}'$ denotes the averaging operator. Defining the nonequilibrium correction factor by $\zeta = \int_0^1 \exp\Big(\big(\frac{\Delta E}{k_{\mathrm{B}}T_{\mathrm{tr}}} - \frac{\Delta E}{k_{\mathrm{B}}T_{\mathrm{in}}}\big)s\Big)\,\mathrm{d}s$ the source term ω_{in} is recast in the convenient form

$$\omega_{\mathrm{in}} = 2\mathfrak{n}^2 \frac{[\![(\Delta E)^2 \zeta]\!]}{k_{\mathrm{B}} T_{\mathrm{tr}} T_{\mathrm{in}}} (T_{\mathrm{tr}} - T_{\mathrm{in}}). \tag{A.7}$$

Defining the nonequilibrium relaxation time by $\tau_{\mathrm{in}} = c_{\mathrm{in}} k_{\mathrm{B}} T_{\mathrm{tr}} T_{\mathrm{in}} / (2\mathfrak{n}[\![(\Delta E)^2\zeta]\!])$, we recover the expression (2.13) of the source term. The Navier-Stokes perturbations of the source terms may also be neglected as discussed in [3].

Letting ϵ be a typical reduced time for internal energy exchanges, the natural scaling associated with the model is thus $\epsilon_d < \epsilon$ since internal energy exchanges are assumed to be slower that the collision time with (A.1) and it is then natural to let $\epsilon_d \to 0$ when $\epsilon \to 0$. It is nevertheless legitimate to also investigate the limiting situation where $\epsilon_d = \epsilon$ starting directly from the fluid model out of equilibrium, and to compare the resulting limit with the traditional one-temperature fluid model obtained with the Chapman-Enskog asymptotics when $\epsilon_d = \epsilon$ [16, 18].

B Generalized-Inverses with prescribed range and nullspace

We summarized in this appendix the definitions and properties of generalized inverses with prescribed range and nullspace [13, 18]. These generalized inverses are often practical and naturally arise in multicomponent transport [11, 13, 18]

Proposition B.1. *Let $G \in \mathbb{R}^{n,n}$ be a matrix, and let $\mathfrak{p}$ and $\mathfrak{q}$ be two subspaces of $\mathbb{R}^n$ such that $N(G) \oplus \mathfrak{p} = \mathbb{R}^n$ and $R(G) \oplus \mathfrak{q} = \mathbb{R}^n$. Then, there exists a unique matrix Z such that $GZG = G$, $ZGZ = Z$, $N(Z) = \mathfrak{q}$, and $R(Z) = \mathfrak{p}$. The matrix Z—termed the generalized inverse of G with prescribed range $\mathfrak{p}$ and nullspace $\mathfrak{q}$—satisfies $GZ = P_{R(G),\mathfrak{q}}$ and $ZG = P_{\mathfrak{p},N(G)}$, where $P_{\mathfrak{a},\mathfrak{b}}$ is defined for linear spaces $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a} \oplus \mathfrak{b} = \mathbb{R}^n$ and denotes the projector onto $\mathfrak{a}$ along $\mathfrak{b}$.*

Proof. We first show that there exists a matrix $\mathcal{J} \in \mathbb{R}^{n,n}$ such that $G\mathcal{J}G = G$ and $\mathcal{J}G\mathcal{J} = \mathcal{J}$. Let e^i, $i = 1, \dots, n$, be a basis of $\mathbb{R}^n$, such that e^i, $i = 1, \dots, r$, is a basis of $N(G)$. Then, by construction, the vectors $\mathsf{f}^i = G\mathsf{e}^i$, $i = r{+}1, \dots, n$ are linearly independent and may be completed to form a basis f^i, $i = 1, \dots, n$ of $\mathbb{R}^n$. Define then $\mathcal{J}$ such that $\mathcal{J}\mathsf{f}^i = 0$, $1 \le i \le r$, and $\mathcal{J}\mathsf{f}^i = \mathsf{e}^i$, $r{+}1 \le i \le n$. One may then easily check that $G\mathcal{J}G\mathsf{e}^i = G\mathsf{e}^i$, $1 \le i \le n$, and $\mathcal{J}G\mathcal{J}\mathsf{f}^i = \mathcal{J}\mathsf{f}^i$, $1 \le i \le n$, so that $G\mathcal{J}G = G$ and $\mathcal{J}G\mathcal{J} = \mathcal{J}$. Defining now $Z = P_{\mathfrak{p},N(G)}\mathcal{J}P_{R(G),\mathfrak{q}}$, we have $GZG = GP_{\mathfrak{p},N(G)}\mathcal{J}P_{R(G),\mathfrak{q}}G = G\mathcal{J}G$, since $G = GP_{\mathfrak{p},N(G)}$ and $P_{R(G),\mathfrak{q}}G = G$, so that $GZG = G$. Similarly, from $ZGZ = P_{\mathfrak{p},N(G)}\mathcal{J}P_{R(G),\mathfrak{q}}GP_{\mathfrak{p},N(G)}\mathcal{J}P_{R(G),\mathfrak{q}}$, we obtain that $ZGZ = P_{\mathfrak{p},N(G)}\mathcal{J}G\mathcal{J}P_{R(G),\mathfrak{q}}$ and thus that $ZGZ = Z$. By construction, we also have $R(Z) \subset \mathfrak{p}$ and $\mathfrak{q} \subset N(Z)$, and, from $GZG = G$ and $ZGZ = Z$, we obtain that $\mathrm{rank}(G) = \mathrm{rank}(Z)$. Since $\dim(\mathfrak{p}) = \mathrm{rank}(G)$ and $\dim(\mathfrak{q}) = n - \mathrm{rank}(G)$ by assumption, we conclude that $R(Z) = \mathfrak{p}$ and $N(Z) = \mathfrak{q}$. From the relations $GZG = G$ and $ZGZ = Z$, we also deduce that GZ and ZG are projectors and that $\mathrm{rank}(GZ) = \mathrm{rank}(G) = \mathrm{rank}(Z) = \mathrm{rank}(ZG)$. Since we also have $R(ZG) \subset R(Z) = \mathfrak{p}$ and $N(G) \subset N(ZG)$, we obtain that $R(ZG) = \mathfrak{p}$ and $N(ZG) = N(G)$ and similarly that $R(GZ) = R(G)$ and $N(GZ) = \mathfrak{q}$, so that $GZ = P_{R(G),\mathfrak{q}}$ and $ZG = P_{\mathfrak{p},N(G)}$. Finally, if there are two such generalized inverses Z_1 and Z_2, we have $Z_i = Z_iGZ_i$, $GZ_i = P_{R(G),\mathfrak{q}}$ and $Z_iG = P_{\mathfrak{p},N(G)}$, $i = 1, 2$, so that $Z_1 = Z_1GZ_1 = P_{\mathfrak{p},N(G)}Z_1 = Z_2GZ_1 = Z_2P_{R(G),\mathfrak{q}} = Z_2GZ_2 = Z_2$. □

Proposition B.2. *Let $\mathcal{E} \subset \mathbb{R}^n$ be a linear subspace of $\mathbb{R}^n$ and $\widetilde{\mathsf{A}}_0$ be a symmetric positive definite matrix. Let G be a symmetric positive semi-definite matrix such that $N(G) = \mathcal{E}$ and let Z be the generalized inverse of G with prescribed nullspace $\widetilde{\mathsf{A}}_0\mathcal{E}$ and range $\widetilde{\mathsf{A}}_0^{-1}\mathcal{E}^\perp = (\widetilde{\mathsf{A}}_0\mathcal{E})^\perp$. Then Z is well defined, Z is symmetric positive semi-definite matrix, and Z depends smoothly on $\widetilde{\mathsf{A}}_0$.*

Moreover, denoting by $\mathsf{e}_1, \dots, \mathsf{e}_{n_\mathrm{e}}$ a basis of the linear subspace $\mathcal{E}$, $\Pi_\mathrm{e} = \mathbb{R}^{n_\mathrm{e}} \longrightarrow \mathbb{R}^n$ the linear operator whose matrices in the canonical bases is $\Pi_\mathrm{e} = [\mathsf{e}_1, \dots, \mathsf{e}_{n_\mathrm{e}}]$, $\overline{\mathcal{J}}_\mathrm{e}$ the matrix of size n_e defined by $\big(\overline{\mathcal{J}}_\mathrm{e}\big)^{-1} = \Pi_\mathrm{e}^t\widetilde{\mathsf{A}}_0\Pi_\mathrm{e}$, then Z may be written in the form

$$Z = \Big(G + \widetilde{\mathsf{A}}_0\Pi_\mathrm{e}\overline{\mathcal{J}}_\mathrm{e}(\widetilde{\mathsf{A}}_0\Pi_\mathrm{e})^t\Big)^{-1} - \Pi_\mathrm{e}\overline{\mathcal{J}}_\mathrm{e}\Pi_\mathrm{e}^t. \tag{B.1}$$

Proof. Since $\widetilde{\mathsf{A}}_0$ is symmetric positive definite we have the direct sum $\widetilde{\mathsf{A}}_0\mathcal{E} \oplus \mathcal{E}^\perp$ as well as $(\widetilde{\mathsf{A}}_0\mathcal{E})^\perp \oplus \mathcal{E}$ and we also have $(\widetilde{\mathsf{A}}_0\mathcal{E})^\perp = \widetilde{\mathsf{A}}_0^{-1}\mathcal{E}^\perp$. Since $N(G) = \mathcal{E}$ by assumptions and $R(G) = \mathcal{E}^\perp$ by symmetry of G, we deduce from Proposition B.1 that there exists a unique generalized inverse Z of G with $N(Z) = \widetilde{\mathsf{A}}_0\mathcal{E}$ and $R(Z) = (\widetilde{\mathsf{A}}_0\mathcal{E})^\perp$. This implies that $N(Z^t) = \big(R(Z)\big)^\perp = N(Z)$ and $R(Z^t) = \big(N(Z)\big)^\perp = R(Z)$ and from the relations $GZG = G$ and $ZGZ = Z$ we next deduce that $GZ^tG = G$ and $Z^tGZ^t = Z^t$ so that $Z = Z^t$ from the uniqueness of generalized inverses with prescribed range and nullspace. Moreover, we have $ZGZ = Z$ so that $\langle Z\mathsf{x}, \mathsf{x}\rangle = \big\langle G(Z\mathsf{x}), Z\mathsf{x}\big\rangle$ and Z is positive semi-definite.

Moreover, it is easily obtained by direct multiplication that $\widetilde{\mathsf{A}}_0\Pi_\mathrm{e}\overline{\mathcal{J}}_\mathrm{e}\Pi_\mathrm{e}^t$ and $\Pi_\mathrm{e}\overline{\mathcal{J}}_\mathrm{e}(\widetilde{\mathsf{A}}_0\Pi_\mathrm{e})^t$ are projector matrices and next that $\widetilde{\mathsf{A}}_0\Pi_\mathrm{e}\overline{\mathcal{J}}_\mathrm{e}\Pi_\mathrm{e}^t = P_{N(Z),R(G)}$ and $\Pi_\mathrm{e}\overline{\mathcal{J}}_\mathrm{e}(\widetilde{\mathsf{A}}_0\Pi_\mathrm{e})^t = P_{N(G),R(Z)}$. From the definition of Z we also have the relations $ZG = P_{R(Z),N(G)}$ and $GZ = P_{R(G),N(Z)}$. Finally performing the product $\big(Z + \Pi_{\rm e}\overline{\mathcal{J}}_{\rm e}\Pi_{\rm e}^t\big)\big(G + \widetilde{\mathsf{A}}_0\Pi_{\rm e}\overline{\mathcal{J}}_{\rm e}(\widetilde{\mathsf{A}}_0\Pi_{\rm e})^t\big)$, and using $Z\widetilde{\mathsf{A}}_0\Pi_{\rm e} = 0$ and $G\Pi_{\rm e} = 0$ it is obtained that

$$\big(Z + \Pi_{\rm e}\overline{\mathcal{J}}_{\rm e}\Pi_{\rm e}^t\big)\big(G + \widetilde{\mathsf{A}}_0\Pi_{\rm e}\overline{\mathcal{J}}_{\rm e}(\widetilde{\mathsf{A}}_0\Pi_{\rm e})^t\big) = P_{R(Z),N(G)} + P_{N(G),R(Z)} = \mathbb{I}_n.$$

This yields (B.1) as well as the smooth dependence of Z on $\widetilde{\mathsf{A}}_0$ and the proof is complete. □

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